HOW FAR CAN WE GO WITH AMITSUR'S THEOREM?

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ABSTRACT. A well-known theorem by S.A. Amitsur shows that the Jacobson radical of the polynomial ring $R[x]$ equals $I[x]$ for some nil ideal I of R. In this paper, however, we show that this is not the case for differential polynomial rings, by proving that there is a ring R which is not nil and a derivation D on R such that the differential polynomial ring $R[x; D]$ is Jacobson radical. We also show that, on the other hand, the Amitsur theorem holds for a differential polynomial ring $R[x; D]$, provided that D is a locally nilpotent derivation and R is an algebra over a field of characteristic $p > 0$.

1. Introduction

Let R be a noncommutative associative ring. In 1956, S.A. Amitsur proved that the Jacobson radical of the polynomial ring $R[x]$ equals $I[x]$ for some nil ideal I of R [14]. Then in 1980, S. S. Bedi and J. Ram extended Amitsur's theorem to skew polynomial rings of automorphism type [5]. The question then arises as to whether Amitsur's theorem also holds for differential polynomial rings; that is, whether the Jacobson radical of $R[x; D]$ equals $I[x; D]$ for a nil ideal I of R. In 1975, D. A. Jordan [12] showed that Amitsur's theorem holds for differential polynomial rings $R[x; D]$, provided that R is a Noetherian ring with an identity, and in 1983, M. Ferrero, K. Kishimoto and K. Motose [8] showed that in the general case the Jacobson radical of $R[x, D]$ equals $I[x, D]$ for an ideal I of R (and I is nil if R is commutative). However, it remained an open question as to whether I needs to be nil. We will answer this question in the negative, by proving the following theorem.

Theorem 1. *There is a ring R which is not nil and a derivation D on R such that the ring $R[x; D]$ is Jacobson radical. Moreover, for an arbitrary prime number p,*

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R can be assumed to be an F-algebra, where F is the algebraic closure of a finite field of p elements.

However, in the case when D is a locally nilpotent derivation we are able to show the following.

Theorem 2. *Let F be a field of characteristic $p > 0$, let R be an F algebra and D be a derivation on R. If D is a locally nilpotent derivation, then the Jacobson radical of the differential polynomial ring $R[x; D]$ equals $I[x]$ for some nil ideal I of R.*

In 1987, J. Bergen, S. Montgomery and D.S. Passman showed that Amitsur's theorem also holds for differential polynomial rings in the case where R is a polynomial identity algebra, and obtained far-reaching related results for enveloping algebras of Lie algebras and crossed products [4]. Surprising applications of derivations in Lie algebras and nil algebras were found by V. M. Petrogradsky, I.P. Shestakov and E. Zelmanov [20, 21, 23]. We also note that the Jacobson radical of a ring $R[x; D]$ in the case when R has no nil ideals was investigated by P. Grzeszczuk and J. Bergen [9]. For other results on such rings, see [17, 28]. Interesting results in the case where R is a polynomial identity ring were obtained by J. Bell, B. Madill and F. Shinko in [3], and by B. Madill in [16]; for example, in [3] it was shown that, if R is a locally nilpotent ring satisfying a polynomial identity, then $R[x; D]$ is Jacobson radical. This does not hold in general for an arbitrary locally nilpotent ring R (see [27]).

The following questions remain open.

Question 1. *Let R be a ring without nil ideals, and D a derivation on R; does it follow then that $R[x; D]$ is semiprimitive?*

Question 2. *Let R be a ring, and D a derivation on R such that $R[x; D]$ is nil. Does it follow that $R[x]$ is nil?*

Let D be a derivation on a ring R. Recall that the differential polynomial ring $R[x; D]$ consists of all polynomials of the form $a_n x^n + \ldots + a_1 x + a_0$, where $a_i \in R$ for $i = 0, 1, 2 \ldots, n$. The ring $R[x; D]$ is considered with pointwise addition and multiplication given by $x^i x^j = x^{i+j}$ and $xa - ax = D(a)$, for all $a \in R$.

For general information on polynomial identity algebras we refer the reader to [6] and [22], and for diferential polynomial rings over associative noncommutative rings to [13] and [7].

We prove Theorem 2 in Section 2. Sections $3-9$ and $10-17$ are mathematically independent of each other and hence can be considered separately (in Sections 3-9 we prove Theorem 1 under the Assumption 1, and in Sections $10-17$ we prove Assumption 1 for algebras over some fields). An outline of the proof for Theorem 1 now follows:

- Let F be a field, and let A' be a free algebra in generators a, b, x, and $\bar{A}$ be the ideal of A' generated by a and b. We introduce ideal I' in $\bar{A}$ which is generated by entries of powers of some matrices $X_1, X_2, \ldots$. It is then shown that $\bar{A}/I'$ is Jacobson radical.
- We introduce ideal $L(I')$ of $\bar{A}$. We define $L(I')$ to be the smallest ideal such that $I' \subseteq L(I')$ and $\gamma_t(I') \subseteq L(I')$ for every $t \in F$, where $\gamma_t : A' \to A'^1$ is the ring homomorphism such that $\gamma_t(a) = a$, $\gamma_t(b) = b$ and $\gamma_t(x) = x + t$.
- It is then shown that $\bar{A}/L(I')$ is isomorphic to some differential polynomial ring $Z[y; D]$.
- Since $\bar{A}/I$ is Jacobson radical, then $\bar{A}/L(I')$ is Jacobson radical. It follows that $Z[y; D]$ is Jacobson radical.
- Next we introduce Assumption 1, and show that if F is a field which is the algebraic closure of a finite field then Assumption 1 holds.
- It is then shown that if Assumption 1 holds then some subrings of $\bar{A}/L(I')$ are not nil, which implies that Z is not nil.

2. Proof of Theorem 2

Let R be a ring. Recall that an element $r \in R$ is quasi-invertible in R if there is $s \in R$ such that $r + s + rs = r + s + sr = 0$. As every ring can be embedded in a ring with an identity element, this can be written as $(1 + r)(1 + s) = 1$. Element s is called a quasi-inverse of r. We start with the following well-known fact

Lemma 3. *Let Q be a ring, and let $a \in Q$ be quasi-invertible, and let $b, c \in Q$ be quasi-inverses of a; then $b = c$.*

Proof. Q is a subring of a ring Q^1 with identity. Then $1+b = (1+b)((1+a)(1+c)) = ((1 + b)(1 + a))(1 + c) = 1 + c$, so $b = c$. $\square$

Let F be a field of characteristic $p > 0$ and let R be an F-algebra. Let D be a locally nilpotent derivation on R. Let $a \in R$, then $D^n(a) = 0$ for some n. Observe that using rule $x \cdot D^n(a) - D^n(a) \cdot x = D^{n+1}(a)$, it can be proved by induction that $D^n(a) = \sum_{i=0}^{n} \alpha_i x^{n-i} a x^i$ where $\alpha_i = (-1)^i n!/i!(n-i)!$ (it can also be inferred

using rule $(p-q)^n = \sum_{i=0}^{n} \alpha_i p^{j-i} q^i$, where p denotes multiplication from the left by x, and q multiplication from the right by x). Then $D^{p^m}(a) = x^{p^m} \cdot a - a \cdot x^{p^m}$.

Proof of Theorem 2: Let notation be as above, and let $a \in R$. D is a locally nilpotent derivation, so there is m such that $0 = D^{p^m}(a) = x^{p^m} \cdot a - a \cdot x^{p^m}$. If $R[x;D]$ is Jacobson radical, then ax^{p^m} is quasi-invertible in $R[x,D]$. Let s be the quasi-inverse of ax^{p^m}; then $s = \sum_{i=0}^{n} a_i x^i$ for some $a_i \in R$. Let S be a subring of R generated by elements $a, a_0, a_1, \ldots, a_n$ and elements $D^i(a)$, $D^i(a_0), D^i(a_1) \ldots, D^i(a_n)$ for $i = 1, 2, \ldots$. Then D is a derivation on S and $S[x;D]$ is a subring of $R[x;D]$. Notice that element ax^{p^m} is quasi-invertible in $S[x;D]$. Recall that D is a locally nilpotent derivation, so there is $k > m$ such that $0 = D^{p^k}(a_i) = x^{p^k} a_i - a_i x^{p^k}$ for $0 \leq i \leq n$. Then x^{p^k} commutes with all elements of S, since $x^{p^k} D^j(a_i) - D^j(a_i) x^{p^k} = D^{p^k + j}(a_i) = 0$. Therefore, D is a nilpotent derivation on S, since $D^{p^k}(s) = x^{p^k} \cdot s - s \cdot x^{p^k} = 0$ for every $s \in S$. Notice that $S[x;D]$ is a subring of a ring Q, where Q is the set of all series $\sum_{i=0}^{\infty} c_i x^i$ with $c_i \in S$ with natural addition and multiplication $xc - cx = D(c)$ for $c \in S$. The multiplication on Q is well defined because D is a nilpotent derivation on S.

Recall that $x^{p^m} \cdot a - a \cdot x^{p^m} = 0$, hence $(ax^{p^m})^i = a^i \cdot x^{i \cdot p^m}$. Observe that for $c = a \cdot x^{p^m}$ we have $(1+c)(1-c+c^2-c^3+\ldots) = 1$. Therefore, $a' = \sum_{i=1}^{\infty} (-1)^i c^i = \sum_{i=1}^{\infty} (-1)^i a^i x^{i \cdot p^m}$ is a quasi-inverse of $a \cdot x^{p^m}$ in Q. By Lemma 3, we get $s = a'$, hence $a' \in S[x;D]$. It follows that $a^i x^{i \cdot p^m} = 0$ for almost all i; hence a is nilpotent.

3. Definitions and the Jacobson radical

Let F be a field. Throughout this paper we will assume that F is a countable field. Notice in particular that the algebraic closure of any finite field is countable. Let A' be a free noncommutative F- algebra generated by elements a, b and x. We assign gradation 1 to elements a and b and we assign gradation 0 to element x. By R we denote the subalgebra of A' generated by a and b, and by A we denote a subalgebra of A' generated by ax^i, bx^i for $i = 0, 1, 2, \ldots$. Notice that $A = RA' + R$, and hence A is a left ideal in A'. By $A'(n)$ we will denote the linear space spanned by all elements with gradation n in A'. In general, if T is a linear subspace of A', then we denote $T(n) = T \cap A'(n)$. In particular, $A(n)$ denotes the linear space spanned by all elements with gradation n in A. For a given ring Q we denote by Q^1 the usual extension with an identity of the ring Q.

Recall that an element $r \in A$ is quasi-invertible in A if there is $s \in A$ such that $r + s + rs = r + s + sr = 0$. As every ring can be embedded in a ring with

an identity element this can be written as $(1+r)(1+s)=1$, and s is called the quasi-inverse of r. By $< x >$ we will denote the ideal generated by x in A'.

The following Lemma is a reformulation of Lemma 7.2 in [26].

Lemma 4. *Let $r \in A$. Then there is matrix X_r of some finite size with entries in $A(1)$ and such that for every $n > 0$, $r + Q_{r,n}$ is quasi-invertible in algebra $A/Q_{r,n}$ where $Q_{r,n}$ is the ideal generated by coefficients of matrix X_r^n in A. If $r \in R$, then the quasi-inverse of r is in R. If $r \in < x >$, then there is $\alpha(X_r)$ such that ${X_r}^i$ has all entries in $< x >$ for every $i > \alpha(X_r)$.*

Proof. To every $r \in A$ we can assign matrix X_r of some finite size with entries in $A(1)$, like in Definition 7.1 in [26]. Let n be a natural number. We can apply Lemma 7.2 from [26] to $S = A(1)$ and $r = \sum_{i=1}^{\gamma} s_i$ with $s_i \in S^i = A(i)$. Recall that we used the following notation in [26], $v_0 = 1$ and v_i is the sum of all products $s_{j_1} s_{j_2} \dots s_{j_k}$ where $j_1 + \dots + j_k = i$ and k is arbitrary. Observe now that by Lemma 7.2 in [26] r is quasiregular in $A/Q(r,n)$.

Observe also that if $r \in A \cap < x >$, then for a sufficiently large n matrix X_r^n has all entries from $A \cap < x >$ by the Remark on page 925 in [26]. Notice also that all coefficients of X_r^n are from $A(n)$.

If $r \in R$, then the entries of matrix $Q_{r,n}$ are in R, hence by the above reasoning applied to ring R instead of A we get that $r + Q_n$ is quasi-invertible in R/Q_n, where Q_n is the ideal generated by entries of ${X_r}^n$ in R. Observe that $Q_n \subseteq Q_{r,n}$. By Lemma 3, $r + Q_{r,n}$ has a quasinverse of the form $s + Q_{r,n}$ for some $s \in R$, as required. □

The field F is countable, hence the set of elements of A is countable (since A is finitely generated). It follows that the set of all matrices X_r for $r \in A'$ is countable. We can ennumerate the matrices X_r with either $r \in A \cap A' x A'$ or $r \in R$ as $X_1, X_2, \dots$.

The main result of this section is the following:

Theorem 5. *Let notation be as above, in particular let matrices $X_1, X_2, \dots$ be as above. Let $0 < m_1 < m_2 < \dots$ be a sequence of natural numbers such that $20m_i$ divides m_{i+1}, and $m_i > \alpha(X_i)$ (where $\alpha(X_i)$ is as in Lemma 4). Let S_i' be the linear space spanned by all entries of the matrix $X_i^{m_i}$ and let*

$$S_i = \sum_{j=1}^{\infty} A(j \cdot 20m_i - 2m_i) S_i' A(m_i) A^1.$$

Then there is an homogeneous ideal I in A contained in $\sum_{i=1}^{\infty} S_i$ and such that A/I is Jacobson radical. Moreover, I is a left ideal in A', and if $g+h \in I$ and $g \in R$ and $h \in A\cap < x >$ then $g, h \in I$.

Proof. Observe first that the ideal I of A generated by entries of the matrices $X_k^{30m_k}$ is contained in the subspace S_k. It follows because entries of every matrix X_k have degree one. Namely, if $n > i+2m_k$ then every entry of matrix X_k^n belongs to $A(i)S_k'A(m_k)A$ for every $0 \leq i$. Observe also that, by Lemma 4, all elements $r \in R$ and all elements $r \in A\cap < x >$ are quasiregular in A/I. Therefore, by Lemma 4 all $r \in R$ and all $r \in A\cap < x >$ are quasiredular in A/I.

Notice also that I is a left ideal in A', as $A(m_k)A' \subseteq A(m_k) + A(m_k)A$.

We will now show that for every $r \in A$ element $r+I$ is quasi-invertible in A/I. Let $r = u+v$, where $u \in R$ and $v \in A\cap < x >$. Since $u \in R$ then by Lemma 4, there is $u' \in R$ such that $(1+u)(1+u')+I = 1+I$. Notice that element $(1+r)+I$ has right inverse if and only if element $(1+r)(1+u')+I$ has right inverse in $(A/I)^1$. We see that $(1+r)(1+u')+I = (1+u+v)(1+u')+I = 1+v(1+u')$. By assumption $1+v(1+u')+I$ has right inverse by Lemma 4, because $v(1+u') \in A\cap < x >$ since $v \in A\cap < x >$. It follows that $1+r+I$ has a right inverse in $(A/I)^1$. In a similar way we show that $1+r+I$ has a left inverse in $(A/I)^1$. Therefore $r+I$ is quasi-invertible in A/I (similarly as in Lemma 3).

The last assertion from the thesis of our theorems follws because $m_i > \alpha(X_i)$, and so the ideal generated by entries of matrix $X_i^{30m_i}$ is either contained in $< x >$ or is generated by elements from R. □

Let $\bar{A} = A + xA + x^2A + \ldots$; then $\bar{A}$ is an F-algebra. Notice that $A' = \bar{A} + F[x]$ where $F[x]$ is the polynomial ring over F.

Lemma 6. *Let $\bar{A}$ be as above. Let I be an ideal in A which is also a right ideal in A' (so $IA' \subseteq I$). Let $I' = I + xI + x^2I + \ldots$. Then I' is an ideal in $\bar{A}$ and $I \cap R = I' \cap R$. In addition if $r+I$ is not a nilpotent in A/I for some $r \in R$, then $r+I'$ is not a nilpotent in $\bar{A}/I'$. Moreover, if A/I is Jacobson radical then $\bar{A}/I'$ is Jacobson radical.*

Proof. From Lemma 4.1 on page 50 in [14] (by interchanging the left and the right side), we see that an element y is in the Jacobson radical of $\bar{A}/I'$ if $(1+yq)+I'$ is right invertible in $\bar{A}/I'$ for every $q \in A'$. Clearly, if $y \in A$ then $yq \in A$ for every $q \in A'$. Observe that A' is a two-sided ideal in A'. By assumption, if $r = yq \in A$ then $(1+r)+I$ is right invertible in A/I and hence $(1+r)+I'$ is right invertible in A'/I'. Therefore every element $a+I'$ for $a \in A$ is in the Jacobson radical of

A'/I'. Recall that the Jacobson radical is a two sided ideal. Therefore, $a+I'$ is in the Jacobson radical of A'/I'. Therefore every element $a'+I'$ for $a'\in\bar{A}$ is in the Jacobson radical of A'/I'. Observe that if $b+I'$ is a quasi-inverse of $a'+I'$, then $b=-a'b-a'\in\bar{A}+I'\subseteq\bar{A}$. Therefore we can assume that $b\in\bar{A}$. It follows that $\bar{A}/I'$ is Jacobson radical. Therefore $\bar{A}/I$ is Jacobson radical.

We will now show that $I\cap R=I'\cap R$. Let $T_i=x^i(RA'+R)=x^iA$ for $i=0,1,2,\ldots$. Recall that A' is a free algebra, and $x\notin R$. Therefore if $0\neq t_i\in T_i$ for $i=0,1,2,\ldots$ then elements $t_0,t_1,\ldots$ are linearly independent over F.

Let $i\in I$, then $i=i_0+i_1+\ldots+i_n$ for some $i_0\in I, i_1\in xI,\ldots,i_n\in x^nI$. Observe that since $I\subseteq A$ then $i_j\in T_j$ for $j=1,2,\ldots,n$. If $i\in R$ then $i-i_0=i_1+i_2+\ldots+i_n$. Notice that $i-i_0\in T_0$. The above observation on elements t_i implies that elements $i-i_0,i_1,i_2,\ldots,i_n$ are all equal zero, so $i=i_0\in I$. Therefore $I\cap R=I'\cap R$.

Suppose now that $r+I'$ is nilpotent in $\bar{A}/I'$. Then $r^n\in I'$ for some n, so $r^n\in I$ by the above, and so $r+I$ is nilpotent in A/I. □

4. Platinum ideals and platinum subspaces

In this section we introduce platinum spaces, which will be useful for constructing examples of differential polynomial rings. Let notation be as in the previous sections, in particular A' is generated by elements a,b and x, and R is generated by elements a and b.

Definition 1 Let P be the smallest subring of A' satisfying the following properties.

- $R\subseteq P$
- If $c\in P$ then $xc-cx\in P$

For a $c\in R$ define $D(c)=xc-cx$. Then D is a derivation on P.

Therefore we can consider the differential polynomial ring $P[y;D]$ where $yc-yc=D(c)$ for $c\in P$. Observe that P can in a natural way be embedded in A'.

Recall that A' is a free algebra with free generators a,b,x. Let $q\in F$ then let $\gamma_q: A'\to A'^1$ be a ring automorphism such that

$$\gamma_q(a)=a,\gamma_q(b)=b,\gamma_q(x)=x+q.$$

Lemma 7. *Let $q\in F$, then $\gamma_q(p)=p$ for every $p\in P$.*

Proof. We proceed by induction using the definition of P. Observe that $\gamma_q(r)=r$ for every $r\in R$. If $u,v\in P$ and $\gamma_q(u)=u$ and $\gamma_q(v)=v$ then $\gamma_q(u+v)=u+v$ and $\gamma_q(uv)=uv$ and $\gamma_q(xu-xu)=(x+q)\gamma_q(u)-\gamma_q(u)(x+q)=xu-ux$. □

Lemma 8. *Let $f : P[y;D] \to A'$ be a F-linear mapping such that $f(p) = p$ and $f(py^i) = px^i$ for $p \in P$, $i = 1, 2, \ldots$. Then f is injective and f is a homomorphism of rings.*

Proof. Observe that P embeds into A' in a natural way as a subring, hence $f(p)$ is well defined as a ring homomorphism for $p \in P$. Every element of $P[y;D]$ can be uniquely written as a linear combination of elements py^i with $p \in P$, hence f is well defined as a linear mapping. We will show that f is a ring homomorphism. Observe that if $p, q \in P[y;D]$ then $py^nq = p\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} D^i(q)y^{n-i}$.

Therefore $f(py^n \cdot qy^j) = f(p\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} D^i(q)y^{n-i}y^j) = p\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} D^i(q)x^{n-i+j}$. On the other hand, $f(py^n)f(qy^j) = px^n \cdot qx^j = p\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} D^i(q)x^{n-i+j}$. Consequently, $f(py^n \cdot qy^j) = f(py^n)f(qy^j)$.

We need to show that the kernel of f is zero. Suppose that $f(\sum_{i=0}^{n} c_iy^i) = 0$ for some $c_i \in P$. Since $f(\sum_{i=0}^{n} c_iy^i) = 0$ then $\sum_{i=0}^{n} c_ix^i = 0$.

If $c = 0$ in A' then clearly $\gamma_q(c) = 0$ for every $q \in F$ (since γ_q is an homomorphism of rings). By Lemma 7 we get $0 = \gamma_q(\sum_{i=0}^{n} c_ix^i) = \sum_{i=0}^{n} c_i(x+q)^i$. We can write such equations for different elements $q_1, q_2, \ldots, q_{n+1} \in F$ and then write them as $(c_0, c_1, \ldots, c_n)M = 0$ where M is a matrix with i-th column equal to $(1, (x+q_i), (x+q_i)^2, \ldots, (x+q_i)^n)^T$. Observe that M is a transposition of a Vandermonde matrix, and hence the determinant of M is $det(M) = \prod_{i>j}(q_i - q_j) \in F$. Hence there is matrix N in $F[x]$ such that $MN = Id \cdot det(M)$, where Id is the identity matrix. It follows that $(c_0, c_1, \ldots, c_n)M = 0$ implies $(c_0, c_1, \ldots, c_n)det(M) = 0$, and so $c_0 = c_1 = \ldots = c_n = 0$. $\square$

Definition Let I be an ideal in A'. We will say that I is a platinum ideal if $\gamma_q(I) \subseteq I$ for every $q \in F$.

Let I be an ideal in A' and let $\bar{I} = I \cap P$. Observe that $\bar{I}$ is an ideal in P and if $c \in \bar{I}$ then $xc - cx \in \bar{I}$ (because $xc, cx \in I$).

Let I be a platinum ideal in A'. Denote $\bar{I} = I \cap P$. For $p \in P$ we define $D(p) = xp - px$. Then $P/\bar{I}$ is a ring with the derivation $D(c+\bar{I}) = D(c) + \bar{I}$ where $D(c) = xc - cx$. Moreover, ring $P/\bar{I}$ can be embedded in A'/I via the mapping $h : P/\bar{I} \to A'/I$, where $h(c + \bar{I}) = c + I$ for $c \in P$ (see [12]).

Theorem 9. *Let I be a platinum ideal in A'. Denote $\bar{I} = I \cap P$. Let $P^* = P/\bar{I}[y;D]$ be the differential polynomial ring with $y(c+\bar{I}) - (c+\bar{I})y = D(c+\bar{I})$ where $D(c+\bar{I}) = xc - cx + \bar{I}$, for $c \in P$. Then P^* can be embedded into A'/I. Moreover, the mapping $f : P^* \to A'/I$ given by $f(p+\bar{I}) = p + I$ for $p \in P$, and $f(py^i + \bar{I}) = px^i + I$ is an injective homomorphism of rings.*

Proof. By the remark before this theorem, f is well defined as a ring homomorphism on $P/\bar{I}$. Every element of P^* can be uniquely written as a linear combination of elements py^i with $p \in P$, $i \geq 0$, so f is well defined as a linear mapping on P^*. To check that f is a ring homomorphism we proceed similarly as in Lemma 38.

We need to show that the kernel of f is zero. Suppose that $f(\sum_{i=0}^{n}(c_i+\bar{I})y^i) = 0$ where $c_i + \bar{I} \in P/\bar{I}$, $c_i \in P$. By the definition of f we have $\sum_{i=0}^{n}(c_i x^i + I) = 0 + I$ in A'/I, hence $\sum_{i=0}^{n} c_i x^i \in I$. Since I is a platinum ideal we get that $\gamma_q(\sum_{i=0}^{n} c_i x^i) \in I$ for every $q \in F$. By Lemma 4, that implies $\sum_{i=0}^{n} c_i(x+q)^i \in I$. Write such equations for different elements $q_1, q_2, \dots, q_{n+1} \in F$, and then write them as $(c_0, c_1, \dots, c_{n+1})M = Q$ where M is a matrix with i-th column equal to $(1, (x+q_i), (x+q_i)^2, \dots, (x+q_i)^n)^T$ and Q is a vector with all entries from I. Observe that M is a transposition of a Vandermonde matrix and hence the determinant of M is $det(M) = \prod_{i>j}(q_i - q_j) \in F$. Hence there is matrix N in $F[x]$ such that $MN = Id \cdot det(M)$ where Id is the identity matrix. It follows that $(c_0, c_1, \dots, c_n)M = Q$ implies $(c_0, c_1, \dots, c_n) \cdot det(M) = QN$. Since $det(M) \in F$ and QN is a vector with all entries in I, then $c_0, c_1, \dots, c_n \in I$. Since $c_i \in P$ then $c_i \in P \cap I = \bar{I}$ for $i = 0, 1, 2, \dots, n$. Therefore $c_i + \bar{I} = 0 + \bar{I}$ for every $i \leq n+1$, and so $\sum_{i=0}^{n}(c_i + \bar{I})y^i = 0$, as required. □

Definition Let S be a linear subspace in A'. We will say that S is a platinum subspace if $\gamma_q(S) \subseteq S$ for every $q \in F$.

Lemma 10. *Given a linear space S, denote $L(S)$ to be the linear space spanned by all elements $\gamma_t(s)$ for $t \in F$, $s \in S$. Then $L(S)$ is the smallest platinum space containing S.*

Proof. If $S \subseteq Q$ and Q is a platinum space then $\gamma_t(S) \subseteq Q$ for every $t \in F$. Therefore $L(S) \subseteq Q$. We need to show that $L(S)$ is a platinum space. Let $s \in L(S)$, then $s = \sum_{t \in W} \gamma_t(s_t)$ where W is a finite subset of F and $s_t \in S$. Let $k \in F$, then $\gamma_k(s) = \sum_{t \in W} \gamma_k(\gamma_t(s_t)) = \sum_{t \in W} \gamma_{k+t}(s_t) \in L(S)$. □

Recall that $\bar{A} = A + xA + x^2A + \dots$.

Lemma 11. *Every element of $\bar{A}$ can be uniquely written in the form $P + Px + Px^2 + \dots$. As $A = RA'$, then every element of A can be written as $RP + RPx + RPx^2 + \dots$. Notice also that $RP \subseteq P \cap A$.*

Proof. It can be shown by induction on n that $x^n P \subseteq P + Px + Px^2 + \dots$. Next observe that the set $P + Px + Px^2 + \dots$ is closed under multiplication and addition,

hence it is a subring of A' containing x^iR and Rx^i for every i, hence it contains $\bar{A}$.

Suppose now that some elements from $P+Px+Px^2+\dots$ are linearly dependent over F; then $\sum_{i=0}^{n} p_i x^i = 0$ for some $p_i \in P$. As A' is a free algebra and γ_t is a ring homomorphism for $t \in F$ then $\gamma_t(\sum_{i=0}^{n} p_i x^i) = 0$. By Lemma 7 we get $\sum_{i=0}^{n} p_i (x+t)^i = 0$. We can write such equations for different elements $t = q_1, t = q_2, \dots, t = q_{n+1} \in F$ and then write these equations as $(p_0, p_1, \dots, p_n)M = 0$, where M is a matrix with i-th column equal to $(1, (x+q_i), (x+q_i)^2, \dots, (x+q_i)^n)^T$. Notice that M is a transposition of a Vandermonde matrix and the determinant of M is $det(M) = \prod_{i>j}(q_i - q_j) \in F$. Therefore M is invertible, with $MN = I$ for some matrix N with entries in $F[x]$. Hence $(p_0, \dots, p_n)M = 0$ implies $(p_0, \dots, p_n)det(M) = 0$, so $p_0 = p_1 = \dots = p_n = 0$, as required. □

Lemma 12. *Let $S \subseteq A'$ be a platinum space. Suppose that $sx \in S$ for every $s \in S$. Then $S = S' + S'x + S'x^2 + \dots$ where $S' = P \cap S$.*

Proof. Observe first that $S'x^i \subseteq S$ for every i. We will show that $S \subseteq S' + S'x + S'x^2 + \dots$. Let $r \in S$. By Lemma 11 we have $r = \sum_{i=0}^{n} p_i x^i \in S$ for some $p_i \in P$. We will proceed by induction on n. If $n = 0$ then $r = p_0 \in P \cap S$, as required. Suppose now that $n > 0$ and the result holds for all numbers smaller than n, and we will show that it holds for n. Because A is a platinum subspace, for every $\alpha \in F$ we have $\sum_i p_i (x+\alpha)^i \in S$. Let $d = \sum_i p_i (x+\alpha)^i - r$, then $d \in S$. Observe that $d = \sum_{i=0}^{n}(p_i(x+\alpha)^i - p_i x^i) = \sum_{i=0}^{n-1} d_i x^i$ for some $d_i \in P$. By the inductive assumption $d_0 \in S$, hence $\sum_{i=1}^{n} p_i \alpha^i = d_0 \in P \cap S$. This holds for every $\alpha \in F$. By the Vandermonde matrix argument, we get $p_1, \dots, p_n \in P \cap S$. Moreover, $p_0 = r - \sum_{i=1}^{n} p_i x^i \in S$ and since $p_0 \in P$ then $p_0 \in P \cap S$. □

5. Linear mappings f and G

Let A^* be the subalgebra of A generated by elements ax^iax^j and bx^ibx^j for all $i, j \geq 0$.

Let $B' \subseteq A(2)$ be a linear F- space spanned by elements ax^ibx^j and bx^iax^j for all $i, j \geq 0$. Let $B = \sum_{i=0}^{\infty} A(2i)B'A$. Observe that $A = A^* + B$ and $A^* \cap B = 0$.

By $F[x]$ we will denote the polynomial ring in variable x over F. Given a linear mapping f by $ker(f)$ and $Im(f)$ we denote the kernel and the image of f.

Lemma 13. *Let m be an even number and let $S \subseteq A^*(m)$ be a platinum space such that $sx \in S$ if $s \in P$. Then there is a linear mapping $f : A^*(m) \to A^*(m)$ such that*

1. *$f(s) = 0$ for every $s \in S$, and the kernel of f is S,*
2. *$f(px^i) = f(p)x^i$ for every $p \in A^*(m) \cap P$,*
3. *$f(p) \in P$ for every $p \in A^*(m) \cap P$.*
4. *Moreover, for every $s \in A^*(m)$ and $t \in F$,*

$$f(\gamma_t(s)) = \gamma_t(f(s)).$$

5. *There is a linear space $E \subseteq A^*(m)$ and such that $f(r) = r$ for $r \in E$, and $E \oplus S = A^*(m)$. Moreover $Im(f) \oplus ker(f) = A^*(m)$.*

Proof. By Lemma 12, $S = S' + S'x + \ldots$ where $S' = S \cap P \subseteq A^*(m)$. By Zorn's lemma, there exists a maximal linear subspace Q of $A^*(m) \cap P$ such that $S' \cap Q = 0$. Observe that then $Q + S' = A^*(m) \cap P$ and that Q is a platinum space (as every subspace of P is a platinum space, by Lemma 7). Define $f(r) = 0$ for $r \in S'$ and $f(r) = r$ for $r \in Q$. Observe that $A^*(m) = \sum_{i=0}^{\infty}(A^*(m) \cap P)x^i$ by Lemma 12. Define $f(px^i) = px^i$ for $p \in A^*(m) \cap P$. By Lemma 11, f is a well defined linear mapping. Notice that $E = Q + Qx + Qx^2 + \ldots$ satisfies (5). Observe that $f(s) = 0$ for every $s \in S$. If $r = r_1 + r_2$ with $r_i \in S'$ and $r_2 \in Q + Qx + Qx^2 + \ldots$ then $f(r) = r_2$, so the kernel of f equals S. The image of f is Q, so (5) holds.

We will now show that $f(\gamma_t(s)) = \gamma_t(f(s))$. Let $s \in S$; then $s = \sum_{i=0,1,\ldots} p_i x^i$ for some $p_i \in S'$. Then $f(s) = \sum_i f(p_i)x^i$. Since S is a platinum space then $\gamma_t(s) \in S$ for any $t \in F$. Observe that by the definition of γ_t we get

$$\gamma_t(s) = \sum_i p_i(x+t)^i,$$

since $\gamma_t(p) = p$ for $p \in P$ by Lemma 7. Therefore $f(\gamma_t(s)) = \sum_i f(p_i)(x+t)^i$.

Observe now that

$$\gamma_t(f(s)) = \gamma_t(\sum_i f(p_i)x^i) = \sum_i f(p_i)(x+t)^i$$

(by Lemma 7, since $f(p_i) \in P$). It follows that $f(\gamma_t(s)) = \gamma_t(f(s))$. $\square$

For a matrix M, let $S(M)$ be the linear space spanned by all entries of M, and $L(M)$ be the linear space spanned by all matrices $\gamma_t(M)$ for $t \in F$, where if M has entries $m_{i,j}$ then $\gamma_t(M)$ has respective entries $\gamma_t(m_{i,j})$. Observe that $L(S(M)) = S(L(M))$.

Definition of mapping G

Let m be an even number and let $f : A^*(m) \to A^*(m)$ be a linear mapping satisfying properties (1)–(5) from Lemma 13 (for some space S). Define a linear mapping $G : A^*(10m) \to A^*(10m)$ as follows:

If $v_1, \ldots, v_{10} \in A^*(m)$ are monomials (products of generators) and $v = v_1 v_2 \ldots v_{10}$, then we define

$$G(v) = G(v_1 \ldots v_{10}) = v_1 v_2 \ldots v_9 f(v_{10}).$$

We can extend the mapping G by linearity to all elements of $A^*(10m)$.

For every natural number $j > 0$ we extend the mapping G to the linear mapping $G : A^*(j \cdot 10m) \to A^*(j \cdot 10m)$ in the following way: if $w = w_1 \ldots w_j$ where w_i are monomials and $w_i \in A^*(10m)$, then we define

$$G(w) = G(w_1)G(w_2) \ldots G(w_j).$$

We can then extend the mapping G by linearity to all elements of $A^*(j \cdot 10m)$.

Moreover, we can also extend the mapping G to matrices with entries in $A^*(j \cdot 10m)$, so if M has entries $a_{i,j}$ then $G(M)$ has respectively entries $G(a_{i,j})$. In similar fashion we can extend the mappings f and γ_t to matrices.

Recall that, for a matrix M, $S(M)$ denotes the linear space spanned by all entries of M, and $L(M)$ is the linear space spanned by matrices $\gamma_t(M)$ for $t \in F$.

Lemma 14. *Let m, G be as in the definition of the mapping G above. Let m_i, n be natural numbers such that n divides m, and $10m$ divides m_i. Let M_i be a matrix with entries in $A(n)$; then*

$$L(S(G(M_i{}^{\frac{m_i}{n}}))) = G(L(S(M_i{}^{\frac{m_i}{n}}))).$$

Proof. Recall that $S(M)$ denotes the linear space spanned by entries of matrix M, hence $S(L(M)) = L(S(M))$ and $S(G(M)) = G(S(M))$ for any matrix M with entries in C. Consequently it is sufficient to show that $L(G(M_i{}^{\frac{m_i}{n}})) = G(L(M_i{}^{\frac{m_i}{n}}))$. We will first show that $G(L(M^{10})) = L(G(M^{10}))$ for any M with entries in $A(m)$. Let $t \in F$, then $G(\gamma_t(M^{10})) = G(\gamma_t(M^9)\gamma_t(M)) = (\gamma_t(M))^9 f(\gamma_t(M))$. By assertion (4) from Lemma 13 we have $f(\gamma_t(M)) = \gamma_t(f(M))$. Therefore $G(\gamma_t(M^{10})) = \gamma_t(M^9) f(\gamma_t(M)) = \gamma_t(M^9)\gamma_t(f(M)) = \gamma_t(G(M^{10}))$. Consequently $G(L(M^{10})) = L(G(M^{10}))$, as required.

By the definition of G, for the same M, and for any $t \in F$, and any number k,

$$G(\gamma_t(M^{10k})) = G(\gamma_t(M^{10}))^k = \gamma_t(G(M^{10}))^k = \gamma_t(G(M^{10k})).$$

Therefore, $G(L(M^{10k}) = L(G(M^{10k}))$. The result now follows when we take $M = M_i^{\frac{m}{n}}$ and $k = \frac{m_i}{10m}$ and substitute in the above equation. $\square$

Lemma 15. *Let M be a finite matrix with entries in $A(j) \cap R$ for some j. For almost all n the dimension of the space $R \cap S(L(M^n)) = \bar{S}(M^n)$ is smaller than $\sqrt{n}$.*

Proof. Since all entries of M are in R then $S(L(M^n)) = S(M^n)$. Let M be an m by m matrix, then the dimension of $\bar{S}(M^n)$ is at most m^2, which for sufficiently large n is smaller than $\sqrt{n}$. □

Recall that by $< x >$ we denote the ideal generated by x in A.

6. Supporting lemmas

Let n, m_1 be natural numbers such that $20n$ divides m_1. Let M_1 be a matrix with entries in $A^*(n)$. Let f be a mapping satysfying properties (1)–(5) from Lemma 13 for $m = 2m_1$ and for the space $S = S(L(M_1^{\frac{m_1}{n}}))A^*(m_1)$. We can then define mapping $G : A^*(j \cdot 20m_1) \to A^*(j \cdot 20m_1)$ as in the previous section (for every j).

Recall that algebra A^{*1} is the usual extension of algebra A^* by adding an identity element.

In the next three lemmas we will use the following notation. Let m be a natural number and let $V \subseteq A^*(m)$ be a linear space. Denote

$$E(V, m) = \sum_{j=1}^{\infty} A^*(j \cdot 20m - 2m)VA^*(m)A^{*1}.$$

We begin with the following lemmas:

Lemma 16. *Let j, n, m_1 be natural numbers such that $20n$ divides m_1. Let M_1 be a matrix with coefficients in $A^*(n)$. Let G be defined as at the beginning of this section. Then the kernel of $G : A^*(j \cdot 20m_1) \to A^*(j \cdot 20m_1)$ is equal to $E(V, m_1) \cap A^*(j \cdot 20m_1)$, where $V = S(L(M_1^{\frac{m_1}{n}}))$.*

Proof. By assertion (5) from Lemma 13 applied for $m = 2m_1$ and $S = VA^*(m_1)$, there is a linear space $E \subseteq A^*(2m_1)$ and such that $f(r) = r$ for $r \in E$ and $E \oplus S = A^*(2m_1)$. For $i = 1, 2, \ldots, j$, let $D = \prod_{i=1}^{j} A^*(18m_1)E$ and let

$$T_i = A^*(i \cdot 20m_1 - 2m_1) \cdot V \cdot A^*((j - i) \cdot 20m_1 + m_1).$$

Observe that $D + \sum_{i=1}^{j} T_i = A(j \cdot 20m_1)$. If $r \in T_i$ for some i, then $G(r) = 0$ by the definition of G. Observe that $E(V, m_1) \cap A^*(j \cdot 20m_1) = \sum_{i=1}^{j} T_i$, hence $E(V, m_1) \cap A^*(j \cdot 20m_1)$ is contained in the kernel of G.

Let r be in the kernel of f. Write $r = t + d$, where $t \in \sum_{i=1}^{j} T_i$ and $d \in D$, then by the definition of G, $G(r) = G(t + d) = G(t) + G(d) = G(d) = d$. Recall that r

is in the kernel of f, so $d=0$ and hence $r \in \sum_{i=1}^{j} T_i$. It follows that the kernel of G equals $E(V, m_1) \cap A^*(j \cdot 20m_1)$. $\square$

Lemma 17. *Let n, k be natural numbers with n even, and let $m_1 < m_2 < \ldots < m_k$ be such that $20nm_i$ divides m_{i+1} for all $1 \leq i < k$ and $20n$ divides m_1. Let M_i be matrices with entries in $A^*(n)$ and let $G : A^*(j \cdot 20m_1) \to A^*(j \cdot 20m_1)$ be defined as at the beginning of this section, and let $j = \frac{m_k}{m_1}$. Let $u \in A^*(20m_k)$, and denote $V_i = L(S(M_i^{m_i/n}))$. Then $u \in \sum_{i=1}^{k} E(V_i, m_i)$ if and only if $G(u) \in \sum_{i=2}^{k} G(E(V_i, m_i))$.*

Proof. Suppose that $G(u) \in \sum_{i=2}^{k} G(E(V_i, m_i))$. It follows that $G(u-e) = 0$ for some $e \in \sum_{i=2}^{k} E(V_i, m_i)$. Consequently $u - e \in ker(G)$, and since by Lemma 16 $ker(G) \subseteq E(V_1, m_1)$, it follows that $u \in \sum_{i=1}^{k} E(V_i, m_i)$.

To see the second implication, suppose now that $u \in \sum_{i=1}^{k} E(V_i, m_i)$; then $G(u) \in \sum_{i=1}^{k} G(E(V_i, m_i))$. By Lemma 16, $E(V_1, m_1) \cap A^*(20m_k) \in ker(G)$, hence $G(L(V_1, m_1)) = 0$. It follows that $G(u) \in \sum_{i=2}^{k} G(E(V_i, m_i))$, since all the considered spaces are homogeneous. $\square$

Lemma 18. *Let notation be as in Lemma 17. For $i = 2, 3, \ldots$ denote*

$$M_i' = G(M_i^{\frac{20m_1}{n}}), W_i = L(S(M_i'^{\frac{m_i}{20m_1}})), T_i = G(E(V_i, m_i) \cap A^*(20m_k)).$$

Then for $i \geq 2$ we have $T_i \subseteq E(W_i, m_i) \cap A^*(20m_k) \subseteq T_i + E(V_1, m_1)$.

Proof. Recall that $20m_1$ divides m_i for all $i \geq 2$; hence

$$T_i = A^*(20m_k) \cap \sum_{j=1}^{\infty} G(A^*(j20m_i - 2m_i))G(L(S(M_i^{\frac{m_i}{n}})))G(A^*(m_i))G({A^*}^1).$$

Observe that

$$M_i'^{\frac{m_i}{20m_1}} = [G(M_i^{\frac{20m_1}{n}})]^{\frac{m_i}{20m_1}} = G(M_i^{\frac{m_i}{n}})$$

by the definition of mapping G. Therefore, and by Lemma 14,

$$L(S(M_i'^{\frac{m_i}{20m_1}})) = L(S(G(M_i^{\frac{m_1}{n} \cdot \frac{m_i}{m_1}}))) = G(L(S(M_i^{\frac{m_i}{n}}))).$$

It follows that

$$T_i = A^*(20m_k) \cap \sum_{j=1}^{\infty} G(A^*(j20m_i - 2m_i))(L(S(M_i'^{\frac{m_i}{20m_1}})))G(A^*(m_i))G({A^*}^1).$$

It follows that $T_i \subseteq E(W_i, m_i) \cap A^*(20m_k)$.

We will now show that $E(W_i, m_i) \cap A^*(20m_k) \subseteq T_i + E(V_1, m_1)$. Recall that the mapping G can be defined on $A(j \cdot 20m_1)$ for any j, and that $20m_1$ divides m_i for each $i > 1$. Observe now that by assertion (5) from Lemma 13, $ker(f) + im(f) =$

$A(m)$, where $im(f)$ is the image of f. Therefore, by the construction of mapping G we get that $A^*(j \cdot 20m_1) = (Im(G) + kerG) \cap A^*(j \cdot 20m_1)$ for every j.

It follows that $A^*(m_i) \subseteq G(A^*(m_i)) + ker(G) \cap A^*(m_i)$. By Lemma 16, $A^*(m_i) \subseteq G(A^*(m_i)) + E(V_1, m_1) \cap A^*(m_i)$. Similarly, $A^*(j \cdot 20m_i - 2m_i)) \subseteq G(A^*(j20m_i - 2m_i)) + E(V_1, n_1) \cap A^*(j \cdot 20m_i - 2m_i)$. It follows that $E(W_i, m_i) \cap A^*(20m_k) \subseteq T_i + E(V_1, m_1)$. $\square$

Theorem 19. *Let notation be as in Lemma 18. Let $u \in A(20m_k)$. Then*

$$u \notin \sum_{i=1}^{k} E(V_i, m_i),$$

if and only if

$$G(u) \notin \sum_{i=2}^{k} E(W_i, m_i).$$

Proof. We will first prove implication $\rightarrow$. Assume on the contrary that $u \notin \sum_{i=1}^{k} E(V_i, m_i)$ and $G(u) \in \sum_{i=2}^{k} E(W_i, m_i)$. Observe that by Lemma 18, $E(W_i, m_i) \cap A^*(20m_k) \subseteq \sum_{i=2}^{k} T_i + E(V_1, m_1)$, hence $G(u) \in \sum_{i=2}^{k} T_i + E(V_1, m_1)$. Therefore there is $g \in \sum_{i=2}^{k} T_i$ and $h \in E(V_1, m_1)$ and $G(u) = g + h$. By assertion (5) from Lemma 13 we get that $Im(G) \cap ker(G) = 0$. Since $g(u)$ and g are in $im(G)$ (the image of mapping G) and $h \in E(V_1, m_1)$ is in the kernel of mapping G, then $G(u) - g = h$ implies $G(u) - g = 0$. Therefore $G(u) \in \sum_{i=2}^{k} T_i$, a contradiction with Lemma 17.

We will now prove the other implication. Suppose that $G(u) \notin \sum_{i=2}^{k} E(W_i, m_i)$. By Lemma 18, $G(u) \notin \sum_{i=2}^{k} T_i$, where $T_i = G(E(V_i, m_i) \cap A(20m_k))$. Observe that by Lemma 16, $G(E(V_1, m_1)) = 0$. Since G is a linear mapping it implies $u \notin \sum_{i=1}^{k} E(V_i, m_i)$ (as otherwise we would have $G(u) \in \sum_{i=2}^{k} T_i$.) $\square$

7. Assumption 1

Let A^* be a subalgebra of A generated by elements $ax^i ax^j$ and $bx^i bx^j$ for all $i, j \geq 0$.

Let $B' \subseteq A(2)$ be a linear F- space spanned by elements $ax^i bx^j$ and $bx^i ax^j$ for all $i, j \geq 0$. Let $B = \sum_{i=0}^{\infty} A(2i)B'A$. Observe that $A = A^* + B$ and $A^* \cap B = 0$.

Recall that, for a matrix M, $S(M)$ denotes the linear space spanned by all entries of M, and $L(M) = \sum_{t \in F} \gamma_t(M)$.

Recall that A^* is a subalgebra of F-algebra A, where F is a field.

The following statement will be called Assumption 1 (for F-algebra A).

Assumption 1. Let M be a matrix with entries in $A^*(j)$ for some j, and such that for almost all α matrix M^α has all entries in $< x >$. Then there are infinitely many n, such that the dimension of the space $R \cap S(L(M^n))$ doesn't exceed $\sqrt{n}$.

Remark

Suppose that Assumption 1 holds, and let k be a natural number. We can apply Assumption 1 to matrix M^k to get the following implication of Assumption 1: Let m be a natural number. There are infinitely many n divisible by k such that the dimension of the space $R \cap S(L(M^n))$ is less than $\sqrt{n}$.

Definition Let l, t, m, n be natural numbers and let $r_0, r_1, \ldots, r_t \in A^*$. We define

$$e(n,m)(r_0, r_1, \ldots, r_t) = \sum_{i_1+\ldots+i_m=n} r_{i_1} r_{i_2} \ldots r_{i_m}.$$

The following lemma is known; the proof can be found in [24] for example.

Lemma 20. *Let m, n, t, l be natural numbers, and let let $r_0, r_1, \ldots, r_l \in A^*(t)$. Denote $e(n,m) = e(n,m)(r_0, r_1, \ldots, r_l)$; then for every $k < m$*

$$e(n,m) = \sum_{i=0}^{n} e(i,k)e(n-i, m-k).$$

Lemma 21. *Let l, t, m be natural numbers and let $r_0, r_1, \ldots, r_l \in A^*(t)$. Denote $r'_i = e(i,m)(r_0, r_1, \ldots, r_l)$. Then for every n and every $i \leq mn$,*

$$e(i,n)(r'_0, r'_1, \ldots, r'_{lm}) = e(i, mn)(r_0, r_1, \ldots, r_l).$$

Proof. We will use induction on n. If $n = 1$ then the result is clear. Suppose that $n > 1$ and that the result holds for all numbers smaller than n. By Lemma 20, $e(i, mn)(r_0, r_1, \ldots, r_l) = \sum_{j=0}^{i} e(j, m(n-1))(r_0, r_1, \ldots, r_l) \cdot e(j,m)(r_0, \ldots, r_l)$. By the inductive assumption and by Lemma 20, we get $e(i, mn)(r_0, r_1, \ldots, r_l) = \sum_{j=0}^{i} e(j, n-1)(r'_0, \ldots, r'_{lm}) \cdot e(j,1)(r'_0, \ldots, r'_{lm}) = e(j,n)(r'_0, \ldots, r'_l)$. $\square$

Recall that $< x >$ is the ideal of A' generated by x.

Lemma 22. *Let F be a field, and suppose that Assumption 1 holds for F-algebra A. Let n, m, t be natural numbers such that $20n$ divides m. Let M be a matrix with coefficients in $A^*(n)$. Assume that either all entries of M are in R or for almost all q entries of M^q are in $< x >$. Let $t \geq 1$, $r_0, r_1, \ldots, r_t \in A^*(n) \cap R$, and denote $e(j,k) = e(j,k)(r_0, r_1, \ldots, r_t)$. Then there exists m and j, j' such that*

$$e(j, \frac{20m}{n}) \notin A^*(18m)VA^*(m)$$

and

$$e(j', \frac{20m}{n}) \notin F \cdot e(j, \frac{20m}{n}) + A^*(18m)VA^*(m),$$

where $V = L(S(M^{\frac{m}{n}}))$ *Moreover, if* $c > 0$ *is a natural number then we can assume that* $20nc$ *divides* m.

Proof. Observe that the linear space spanned by $e(i, \frac{20m}{n})$ for $i = 0, 1, 2, \dots$ has dimension $\frac{20m}{n} + 1$, which is larger than $\sqrt{\frac{20m}{n}}$ for almost all m. By Assumption 1 and Lemma 15, there is m and $j, j' \leq m$ such that $e(j, \frac{m}{n}) \notin T$ and $e(j', \frac{m}{n}) \notin T + F \cdot e(j', \frac{m}{n})$, where $T = S(L(M^{\frac{m}{n}}))$. Moreover we can assume that $20nc$ divides m, by the remark after Assumption 1.

Let j be minimal such that $e(j, \frac{m}{n}) \notin T$ and j' minimal such that $e(j', m) \notin T + F \cdot e(j, \frac{m}{n})$. We claim that $e(j, \frac{20m}{n}) \notin A^*(18m)TA(m)$ and $e(j', \frac{20m}{n}) \notin A^*(18m)TA(m) + F \cdot e(j, \frac{20m}{n})$.

Notice that $e(0, \frac{m}{n}) = r_0{}^{\frac{m}{n}}$. Recall that A^* is a graded algebra, and so $e(j, \frac{m}{n})e(0, \frac{m}{n}) \notin TA^*(m_1)$; this can be seen by comparing the elements from $A^*(m)$ at the end of each side. By Lemma 20, $e(j, \frac{2m}{n}) = e(j, \frac{m}{n})e(0, \frac{m}{n}) + \sum_{i<j} e(i, \frac{m}{n})e(j-i, \frac{m}{n})$. By the minimality of j, we get $e(j, \frac{2m}{n}) \notin TA^*(m)$. Notice also that, by a similar argument j is minimal such that $e(j, \frac{2m}{n}) \notin TA^*(m)$. Observe now that $e(0, \frac{18m}{n})e(j, \frac{2m}{n}) \notin A^*(18m)TA^*(m)$; this can be seen by comparing the elements from $A^*(18m)$ at the beginning of each side. By Lemma 20, $e(j, \frac{20m}{n}) = e(0, \frac{18m}{n})e(j, \frac{2m}{n}) + \sum_{i<j} e(j-i, \frac{18m}{n})e(i, \frac{2m}{n})$. Recall that j was minimal such that $e(j, \frac{2m}{n}) \notin TA^*(m)$, therefore $e(j, \frac{20m}{n}) \notin A^*(18m)TA(m)$.

Observe now that, since $e(j', \frac{m}{n}) \notin T + F \cdot e(j, \frac{m)}{n}$, then by the same reasoning as above applied to the set $T' = T + F \cdot e(j, 20m)$ instead of the set T, we get

$$e(j', \frac{20m}{n}) \notin A^*(18m)T'A^*(m).$$

By the definition of T' we have $e(j, \frac{m}{n}) \in T'$ and by the minimality of j, $e(i, \frac{m}{n}) \in T$ for $i < j$. By Lemma 20,

$$e(j, \frac{20m}{n}) \in A^*(18m) \sum_{i=0}^{j} e(i, \frac{m}{n})A^*(m) \subseteq A^*(18m)T'A^*(m).$$

Therefore $A^*(18m)TA(m) + F \cdot e(j, 20m) \subseteq A^*(18m)T'A^*(m)$. Therefore $e(j', \frac{20m}{n}) \notin A^*(18m)TA^*(m) + F \cdot e(j, 20m)$, as required. □

We will now introduce Assumption 2.

Assumption 2 Let n, k be natural numbers with n even, and let $m_1 < m_2 < \dots < m_k$ be such that $20nm_i$ divides m_{i+1} for all $1 \leq i < k$ and $20n$ divides m_1. Let M_i for $i = 1, 2, \dots, k+1$ be matrices with entries in $A^*(n)$ and such that either ${M_i}^q$ has entries in $< x >$ for almost all natural q, or M_i has entries in R. Denote $V_i = L(S(M_i^{m_i/n}))$.

let $G : A^*(t \cdot 20m_1) \to A^*(t \cdot 20m_1)$ be defined as in Lemma 16 for $t = \frac{m_k}{m_1}$.

For $i = 2, 3, \dots$ denote

$$M_i' = G(M_i^{\frac{20m_1}{n}}), W_i = L(S(M_i'^{\frac{m_i}{20m_1}})), T_i = G(E(V_i, m_i) \cap A^*(20m_k)).$$

Lemma 23. *Let notation be as in Assumption 2 and suppose that Assumption 2 holds. Let $k \geq 0$ be a natural number. If $k = 0$ let $r_0, r_1, \dots, r_l \in A^*(n)$ and if $k > 0$ then let $r_0, r_1, \dots, r_l \in R \cap A^*(20m_k)$ for some $l \geq 1$. Suppose that there are j, j' such that*

$$\alpha \cdot r_j + \beta \cdot r_{j'} \notin W,$$

provided that $\alpha, \beta \in F$ are not both zero, where $W = \sum_{i=1}^{k} E(V_i, m_i)$ (for $k = 0$ we have $W = 0$). Denote $e(j, k) = e(j, k)(r_0, r_1, \dots, r_t)$. Then if $k > 0$ then there exist m_{k+1} such that for some l, l',

$$\alpha \cdot e(l, \frac{m_{k+1}}{m_k}) + \beta \cdot e(l', \frac{m_{k+1}}{m_k}) \notin W',$$

provided that $\alpha, \beta \in F$ are not both zero, where $W' = \sum_{i=1}^{k+1} E(V_i, m_i)$ and $V_{k+1} = L(S(M_{k+1}{}^{\frac{m_{k+1}}{n}}))$. Moreover, $20nm_k$ divides m_{k+1}. In the case when $k = 0$ then there exists m_1 such that

$$\alpha \cdot e(l, \frac{20m_1}{n}) + \beta \cdot e(l', \frac{20m_1}{n}) \notin W',$$

provided that $\alpha, \beta \in F$ are not both zero.

Proof. We will proceed by induction on k. If $k = 0$ then the result follows from Lemma 22 applied for $m = m_1$ and matrix $M = M_1$. Let $k \geq 1$ and assume that the thesis is true for all numbers smaller than k; we will prove it for k.

By the assumption there are j, j' such that $\alpha \cdot r_j + \beta \cdot r_{j'} \notin W$ provided that α, β are not both zero. Let f, G be as in Theorems 16 and 19; then by Theorem 19

$$\alpha \cdot G(r_j) + \beta \cdot G(r_{j'}) \notin \sum_{i=2}^{k} E(W_i, m_i),$$

provided that $\alpha, \beta \in F$ are not both zero. Recall that $M_i' = G(M_i)^{\frac{20m_1}{n}}$ and $W_i = L(S(M_i'^{\frac{m_i}{20m_1}}))$ for $i = 2, 3, \dots, k$. If $k = 1$ then we apply Lemma 22 for $n = 20m_1$, $m = m_2$ and matrix $M = G(M_1^{\frac{20m_1}{n}})$. If $k > 1$ then observe that the number of matrices M_i' is $k - 1$, hence we can apply the inductive assumption to matrices $M_2', M_3', \dots, M_k'$ and to the matrix $M_{k+1}' = G(M_{k+1})^{\frac{20m_1}{n}}$ and elements $\bar{r}_i = G(r_i)$ to obtain that there are j, j' such that if $\alpha, \beta \in F$ are not both zero then

$$\alpha \cdot G(e(j, \frac{m_{k+1}}{m_k})) + \beta \cdot G(e(j', \frac{m_{k+1}}{m_k})) \notin \sum_{i=2}^{k+1} E(W_i, m_i),$$

since $G(e(j, \frac{m_{k+1}}{m_k})) = e(j, \frac{m_{k+1}}{m_k})(G(r_0), G(r_1), \dots, G(r_l))$.

By Theorem 19, we get $\alpha \cdot e(j, \frac{m_{k+1}}{m_k}) + \beta \cdot e(j', \frac{m_{k+1}}{m_k})) \notin W'$, unless $\alpha = \beta = 0$. □

Let $X_1, X_2, \dots$ be matrices as in Theorem 5. Let $Y_1, Y_2, \dots$ be matrices such that Y_i has entries in $A^*(2)$ and $X_i^2 = Y_i + M(B')$, where $M(B)$ is the set of matrices with entries in $M(B')$, where B' is defined as at the beginning of this section.

Recall that $e(i,n)(a^2, b^2)$ is the sum of all products i elements b^2 and $n-i$ elements a^2.

Theorem 24. *Let F be a countable field, and suppose that Assumption 1 holds for F-algebra A. For every i, n denote $e(i,n) = e(i,n)(a^2, b^2)$. Let $Y_1, Y_2, \dots$ be as above, then there are natural numbers $m_1 < m_2 < \dots$ such that $200nm_i$ divides m_{i+1} for all i and 40 divides m_1. Moreover, for every n there are j such that $e(j,n) \notin T$, provided that $\alpha, \beta \in F$ are not both zero, where $T = \sum_{i=1}^{\infty} E(V_i, m_i)$ and $V_i = L(S(Y_i^{\frac{m_i}{2}}))$.*

Proof. Observe first that if $u(j,n) \in T$ for some n and all $j \leq n$, then by Lemma 20 for every $n' > n$ we have $e(j, n') \in T$, for all $j \leq 2n'$. Notice that $e(j,n) \subseteq A^*(2n)$ for every n.

Therefore, it is sufficient to prove that there are $m_1 < m_2 < \dots$ and $j_1, j_2, \dots$ and $j_1', j_2', \dots$ such that for every k,

$$\alpha \cdot e(j_k, 10m_k) + \beta \cdot e(j_k', 10m_k) \notin T,$$

provided that $\alpha, \beta \in F$ are not both zero. Notice that

$$T \cap A^*(20m_k) = \sum_{i=1}^{k} E(V_i, m_i),$$

since all spaces $E(V_i, m_i)$ are homogeneous.

We will construct numbers $m_1, m_2, \dots$ inductively using Lemma 23. By Lemma 23, there is m_1 and j, j' such that $\alpha \cdot e(j, 10m_1) + \beta \cdot e(j', 10m_1) \notin T$, provided that $\alpha, \beta \in F$ are not both zero, moreover 40 divides m_1.

Suppose now that for some $k \geq 1$ we constructed elements $m_1, \dots, m_k$ such that if $\alpha, \beta \in F$ are not both zero then

$$\alpha \cdot e(j_k, 10m_k) + \beta \cdot e(j_k', 10m_k) \notin T.$$

By Lemma 23 there are l, l' such that

$$\alpha \cdot e(l, \frac{m_{k+1}}{m_k})(r_0, \dots, r_{10m_k}) + \beta \cdot e(l', \frac{m_{k+1}}{m_k})(r_0, \dots, r_{10m_k}) \notin T,$$

where $r_i = e(i, 10m_k)$. By Lemma 21, we get $e(i, 10m_{k+1}) = e(i, \frac{m_{k+1}}{m_k})(r_0, \ldots, r_{10m_k})$. Consequently, $\alpha \cdot e(l, 10m_{k+1}) + \beta \cdot e(l', 10m_{k+1}) \notin T$, therefore we constructed m_{k+1} satisfying the thesis of our theorem. Continuing in this way we construct all elements m_i. $\square$

8. Nility

Let A^* be a subalgebra of A generated by elements ax^iax^j and bx^ibx^j for all $i, j \geq 0$.

Let $B' \subseteq A(2)$ be a linear F- space spanned by elements ax^ibx^j and bx^iax^j for all $i, j \geq 0$. Let $B = \sum_{i=0}^{\infty} A(2i)B'A$. Observe that $A = A^* + B$ and $A^* \cap B = 0$. In this chapter we denote

$$e(k, n) = e(k, n)(a^2, b^2)$$

to be the sum of all products of k elements b^2 and $n - k$ elements a^2, so $e(k, n) \in A^*(2n)$.

Lemma 25. *Suppose that J is a homogeneous ideal in A such that $R/R \cap J$ is a nil algebra. Then there is $m > 0$ such that $e(k, m) \in J + B$, for every $0 \leq k \leq m$. Moreover, for every $m' > m$ we have $e(k, m') \in J + B$ for every $0 \leq k \leq m'$.*

Proof. By assumption, there is a number m such that for every $n \geq m$ we have $(a + b^2)^n \in J$. Let $v(k, n)$ be the sum of all products of k elements b^2 and $n - 2k$ elements a; then $v(k, n) \in A(n)$. Observe that

$$(a + b^2)^n = \sum_{0 \leq k \leq \frac{n}{2}} v(k, n + k) \in J,$$

and since J is homogeneous it follows that

$$v(k, n + k) \in J$$

for every natural $k \leq n$, and every $n \geq m$.

Therefore $v(k, 2m) = v(k, (2m - k) + k) \in J$ for every $0 \leq k \leq m$, since $2m - k \geq m$. Observe that $e(k, m) - v(k, 2m) \in B$, so $e(k, m) \in B + J$.

The second part of thesis follows if we take any $m' > m$ and apply the same reasoning to m' instead of m. $\square$

Let I be an ideal in A'. Recall that I is a platinum ideal if $\gamma_q(I) \subseteq I$ for every $q \in F$, and that if Q is a subspace of A' then $L(Q) = \sum_{t \in F} \gamma_t(Q)$.

Theorem 26. *Let notation be as in Theorem 5, and denote $Q = \sum_{i=1}^{\infty} S_i$. Then there is a homogeneous ideal J in A which is a platinum ideal, A/J is Jacobson radical and J is contained in $L(Q)$. Moreover, J is a left ideal in A'.*

Proof. By Theorem 5, there is an ideal I in A such that $I \subseteq Q$. Denote $J = L(I)$; then $L(I) = \sum_{t\in F} \gamma_t(I)$. Again by Theorem 5, $L(I) \subseteq L(Q)$. We claim that $L(I)$ is an ideal in A, and a left ideal in A'. We need to show that if $\alpha \in L(Q)$ then $r\alpha, \alpha r' \in L(I)$, for every $r \in A$ and $r' \in A'$. Since $\alpha \in L(Q)$ then $\alpha = \sum_{t\in W} \gamma_t(s_t)$ for some finite subset W of F, and where $s_t \in I$. Observe that $r\gamma_t(s) = \gamma_t(\gamma_{-t}(r)s_t) \in \gamma_{-t}(I) \subseteq L(I)$ and $\gamma_t(s)r' = \gamma_t(s\gamma_{-t}(r')) \subseteq \gamma_t(I) \subseteq L(I)$, since by Theorem 5 I is a left ideal in A'. By Lemma 10, $L(I)$ is a platinum ideal in A. Since A/I is a Jacobson radical then $A/L(I)$ is a Jacobson radical, so we can set $J = L(I)$.

□

9. Assumption 1 implies Theorem 1

Recall that, given matrix M, $S(M)$ denotes the linear space spanned by all entries of M, and $L(M)$ is the linear space spanned by matrices $\gamma_t(M)$ for $t \in F$. Observe that $L(S(M)) = S(L(M))$. For example, $S(L(Y^n))$ denotes the linear space over F spanned by coefficients of all matrices $\gamma_t(Y^n)$ for every $t \in F$. Recall also that R is a subalgebra of A generated by a and b. The aim of this section is to prove the following.

Theorem 27. *Let F be a countable field, and suppose that Assumption* 1 *holds for F-algebra A. Then there is an F-algebra Z and a derivation D on Z such that the differential polynomial ring $Z[y; D]$ is Jacobson radical but Z is not nil.*

Proof. Let $m_1, m_2, \dots$ be as in Theorem 24 and denote $T = \sum_{i=1}^{\infty} E(V_i, m_i)$ and $V_i = L(S(Y_i^{\frac{m_i}{2}}))$. By Theorem 24 for every n there are $j, j' \leq n$ such that $\alpha \cdot e(j, n) + \beta \cdot e(j', n) \notin T$, provided that $\alpha, \beta \in F$ are not both zero.. Observe that since $e(j, n) \in A^*$ and $T \subseteq A^*$ and $A^* \cap B = 0$, it follows that for every n there are $j, j' \leq n$ such that

$$\alpha \cdot e(j, n) + \beta \cdot e(j', n) \notin T + B,$$

provided that $\alpha, \beta \in F$ are not both zero.

By Theorem 26 applied for such $m_1, m_2, \dots$ we get that there is a homogeneous ideal J in A which is a platinum ideal, A/J is Jacobson radical and J is contained in $L(Q)$, where $Q = \sum_{i=1}^{\infty} S_i$ as in Theorem 5. Moreover, J is a left ideal in A'.

Let $J' = J + xJ + x^2J + \ldots$. Let $\bar{A} = A + xA + x^2A + \ldots$; then $\bar{A}$ is an F-algebra and J' is an ideal in $\bar{A}$. By Lemma 6, $\bar{A}/J'$ is Jacobson radical. In addition, if $r + J$ is not a nilpotent in A/J for some $r \in R$, then $r + J'$ is not a nilpotent in $\bar{A}/J'$.

We will now show that $R/R \cap J$ is not a nil algebra. Observe now that $L(S_i) \subseteq \sum_{j=1}^{\infty} A(j \cdot 20m_i - 2m_i)L(S_i')A(m_i)A^1$, where $S_i' = S({X_i}^{m_i})$ (since $L(A(j)) \subseteq A(j)$ for any j), and that $X_i^2 = Y_i + B_i$, where B_i are matrices with entries in B. Moreover $A(j) \subseteq A^*(j) + B$ for every j, by the definition of B. It follows that $L(S_iA^1) \subseteq E(V_i, m_i) + B$ for every i. By the first part of this proof we get that for every n there are $j, j' \leq n$ such that $\alpha \cdot e(j, n) + \beta \cdot e(j', n) \notin T + B$. On the other hand $J \subseteq L(Q) = \sum_{i=1}^{\infty} L(S_i) \subseteq T + B$. It follows that

$$\alpha \cdot e(j, n) + \beta \cdot e(j', n) \notin J$$

provided that $\alpha, \beta \in F$ are not both zero. By Lemma 25 we get that $R/R \cap J$ is not a nil algebra. So there is $r \in R$ such that $r + J$ is not nilpotent in A/J. By Lemma 6, $r + J'$ is not nilpotent in $\bar{A}/J'$.

Let P be as in Theorem 9; so if $a \in P$ then $ax - xa \in P$, $R \subseteq P$ and P is the smallest subring of A' with this property. We can apply Theorem 9 to $I = J'$ and let $Z = P/\bar{J}$ where $\bar{J} = P \cap J'$. Then $Z[y; D]$ is a differential polynomial ring with $y(r + \bar{J}) - (r + \bar{J})y = D(r) + \bar{J}$, where $D(r) = xr - rx + \bar{J}$. By Theorem 9, $Z[y; D]$ can be embedded in A'/J'. By looking more closely at the mapping f in Theorem 38, we see that the image of $Z[y; D]$ equals $\bar{A}/J'$ where $\bar{A} = A + xA + x^2A + \ldots$, hence the image of $Z[y; D]$ is Jacobson radical. Recall that f is injective, hence $Z[y; D]$ is Jacobson radical.

It remains to show that Z is not a nil ring. Let $r \in R$ be such that $r + J$ is not nilpotent in A/J'. It follows that $r^n \notin J'$, and therefore $r^n \notin \bar{J} = J \cap P$ for $n = 1, 2, \ldots$. Therefore $r + \bar{J}$ is not nilpotent in $P/\bar{J}$, so $P/\bar{J}$ is not nil. $\square$

10. Proof of Assumption 1: Constructing a platinum ideal

We will say that the matrix M satisfies **Assumption** 3 if the following holds:

1. $M = \sum_{i=1}^{\xi} A_i a_i$, where for each i, A_i is a matrix with coefficients in F, and $a_1, a_2, \ldots, a_\xi$ are elements from A which are linearly independent over F and have the same degree – say α. Moreover, for some $\alpha > 0$, each $a_i \in A(\alpha) \cap < x >$ (where $< x >$ is the ideal generated by x in A').
2. By H we will denote the F-algebra generated by matrices A_i, and by W we will denote the Wedderburn radical of H, which is the largest nilpotent ideal in H.

3 $A_1 = e$, where e is a matrix such that $e^2 = e$, $r - er \in W$, $r - re \in W$ and s is a natural number such that $W^s = 0$.

Let $A'^1[y]$ be the polynomial ring in variable y over ring A'^1.

Definition For a variable y (commuting with all elements from A') let $\gamma_y : A' \to A'^1[y]$ be a homomorphism of algebras such that $\gamma_y(a) = a, \gamma_y(b) = b$ and $\gamma_y(x) = x + y$.

For a variable y (commuting with all elements from A') we can write

$$\gamma_y(M) = \sum_{i=0}^{t} y^i M_i,$$

where all entries of M_i are homogeneous elements of A of degree α, similarly to the entries of matrix M.

Let matrix $M = \sum_{i=1}^{\xi} A_i a_i$ satisfy Assumption 3. Let $\gamma_y(M) = \sum_{i=0}^{t} y^i M_i$. Denote

$$w(n,m) = \sum_{i_1+i_2+\ldots+i_m=n} M_{i_1} M_{i_2} \ldots M_{i_m}.$$

The following lemma is known, and is another variant of Lemma 20.

Lemma 28. *Let notation be as above. Let m, n be natural numbers; then for every $k < m$*

$$w(n,m) = \sum_{i=0}^{n} w(i,k) w(n-i, m-k).$$

Lemma 29. *Let F be an infinite field. Let matrix $M = \sum_{i=1}^{\xi} A_i a_i$ satisfy Assumption 3 and $\gamma_y(M) = \sum_{i=0}^{t} y^i M_i$. Then for each j,*

$$M_j \subseteq \sum_{i=1}^{\xi} A_i A.$$

Moreover, for every n, m $w(m,n) \subseteq HA$ where H is the algebra generated by matices $A_1, \ldots, A_\xi$.

Proof. Recall that $\gamma_y(M) = \sum_{i=0}^{t} y^i M_i$; therefore by substituting $y = \alpha_i$ for $\alpha \in F$, we get $\gamma_\alpha(M) = \sum_{i=0}^{t} \alpha^i M_i$. By applying this for $\alpha = t_1, \ldots, \alpha = t_\xi$ for various $t_1, \ldots, t_\xi \in F$, and using the fact that a Vandermonde matrix is invertible that each M_i is a linear combination of matrices $\gamma_\alpha(M)$ for various $\alpha \in F$. Recall that $M = \sum_{i=1}^{\xi} A_i a_i$, hence $\gamma_\alpha(M) = \sum_{i=1}^{\xi} A_i \gamma_\alpha(a_i)$. Therefore each M_j is a linear combination of matrices $A_i \gamma_\alpha(a_i) \subseteq A_i A$. By the definition of $w(n,m)$ we get $w(n,m) \in HA$. $\square$

Recall that, given matrix M, $S(M)$ denotes the linear space spanned by all entries of M, and $L(M)$ is the linear space spanned by matrices $\gamma_t(M)$. Observe that $L(S(M)) = S(L(M))$.

Lemma 30. *Let F be an infinite field and let n be a natural number. Let Q be the linear subspace of A spanned by all entries of matrices $w(n,m)$ for $n = 0, 1, \ldots$; then all entries of the matrix M^m belong to Q. Moreover $Q = S(L(M^m))$.*

Proof. By assumption $\gamma_y(M^m) = (\sum_{i=0}^{t} y^i M_i)^m$, hence for any $\alpha \in F$, when substituting $y = \alpha$ we get

$$\gamma_\alpha(M^m) = (\sum_{i=0}^{t} \alpha^i M_i)^m = \sum_{i=0}^{m\cdot t} \alpha^i w(i,m).$$

Therefore every entry of a matrix $\gamma_\alpha(M^m)$ is in the subspace generated by entries of matrices $w(i,n)$ for various i; hence $S(L(M^m) \subseteq Q$.

On the other hand, let $\alpha_1, \alpha_2, \ldots, \alpha_{tm}$ be different elements of F, then we can substitute $\alpha = \alpha_j$ into the equation

$$\sum_{i=0}^{t\cdot m} \alpha^i \cdot w(i,m) = \gamma_\alpha(M^m).$$

We can then use the Vandermonde matrix argument to show that each $w(i,m)$ is a linear combination of elements $\gamma_{\alpha_i}(M^m)$ for $1 \leq i \leq tm$. Therefore $Q \subseteq S(L(M^m))$. $\square$

11. Embedding platinum subspaces in bigger subspaces

Let notation be as in the previous section. Let M be a matrix satisfying Assumption 3. Let H, W, e be as in Assumption 3.

Definition

Recall that H can be considered as a subalgebra of some matrix algebra over F. Let H' denote algebra $H + KI$, where I is the identity matrix.

Recall that

$$w(n,m) = \sum_{i_1+i_2+\ldots+i_m=n} M_{i_1} M_{i_2} M_{i_3} \ldots M_{i_m}.$$

Let s be such that $W^s = 0$.

Let $f_1, f_2, \ldots$ be elements from H'. We define element $(f_1, f_2, \ldots) * w(m,n)$ for $j = 1, 2, \ldots$ in the following way:

$$(f_1, f_2, \ldots) * w(n,m) = \sum_{i_1+i_2+\ldots+i_m=n} f_1 M_{i_1} f_2 M_{i_2} f_3 M_{i_3} \ldots f_m M_{i_m}.$$

Lemma 31. *Let notation be as above and let m be a natural number larger than s (where $W^s = 0$). Let $u = (f_1, f_2, \ldots)$ and $f_1, f_2, \ldots, f_s \in W \cup \{1 - e\}$; then $u * w(n, m) = 0$, for every n.*

Proof. Observe that for every k, $f_k M_{i_k}$ is a matrix with entries in W; hence $u * w(n, m)$ has entries in $W^s = 0$. □

Definition For $n \geq 1$ define $t_n = (f_1, f_2, \ldots)$, where $f_1 = f_2 = \ldots = f_{n-1} = 1-e$, $f_n = e$ and $I = f_{n+1} = f_{n+2} = \ldots$. For $n \geq 0$ define $t'_n = (f_1, f_2, \ldots)$, where $f_1 = f_2 = \ldots = f_n = 1 - e$ and $I = f_{n+1} = f_{n+2} = \ldots$

Lemma 32. *Let notation be as above. Then for every m, n,*

$$w(n, m) = \sum_{i=0}^{s} t_i * w(n, m).$$

Proof. Observe first that $w(n, m) = (I, I, \ldots) * w(n, m) = t_1 * w(n, m) + t'_1 * w(n, m)$; then $t'_1 * w(n, m) = t_2 * w(n, m) + t'_2 * w(n, m)$, and for every i, $t'_i * w(n, m) = t_{i+1} * w(n, m) + t'_{i+1} * w(n, m)$. By summing all these equations for $i = 1, 2, \ldots, s$, and the equation $w(n, m) = t_1 * w(n, m) + t'_1 * w(n, m)$, we get $w(n, m) + \sum_{i=1}^{s} t'_i * w(n, m) = \sum_{i=1}^{s+1} t_i * w(n, m) + t'_i * w(n, m)$. By Lemma 31, $t_{s+1} = t'_{s+1} = t_s = 0$. It follows that $w(n, m) = \sum_{i=0}^{s+1} t_i * w(n, m)$. □

Definition of set $D(M)$

Let matrix M satisfy Assumption 3, and let H, W, e be as in Assumption 3.

Fix $E_1, E_2, \ldots, E_\beta$ - a basis of W - the Wedderburn radical of P for some β.

Let $E = \{1 - e, E_1, E_2, \ldots, E_\beta\}$.

Let $P_1, P_2, \ldots, P_{\beta'}$ be such that $E_1, \ldots, E_\beta, P_1, \ldots, P_{\beta'}$ span algebra H. We can assume that for every i

$$P_i e = P_i.$$

It follows because $H = He + H(1 - e) \subseteq He + W$. Moreover $e = he$ for $h = e$.

Recall that $W^s = 0$.

Let $0 \leq q < s$. We say that element $(f_1, f_2, \ldots)$ is good and has distance $q + 1$ if $f_1, \ldots, f_q \in E$, $f_{q+1} \in \{e, P_1, P_2, \ldots, P_\beta\}$ and $I = f_{q+2} = f_{q+3} = \ldots$.

The set of all good elements will be denoted $D(M)$.

Lemma 33. *Let M be a matrix satisfying Assumption 3. The set $D(M)$ is finite.*

Proof. It follows because every element in $D(M)$ has distance at most s. □

Lemma 34. *Let M be a matrix satisfying Assumption 3. Then, for every m, n, $w(n, m)$ is a linear combination of elements $u * w(n, m)$ with $u \in D(M)$.*

Proof. It follows from Lemma 32, since every $t_i \in D(M)$. □

Fix m. Let $u \in D$. Recall that $u * w(n,m)$ is a matrix with coefficients in A. By $u * w(n,m)_{k,l}$ we denote the element of A which is at the k,l entry of matrix $u * w(n,m)$. By $[u * w(n,m)]_{k,l}$ we will mean the quintuple (u,n,m,k,l).

Definition of ordering

We first denote an ordering on elements of $E : E_1 < E_2 < \ldots < E_\beta$. We then define $E_\beta < Id - e < e < P_1$ and $P_1 < P_2 < \ldots < P_{\beta'}$. We can now define a lexicographical ordering on the good set $D(M)$.

In particular, if the distance of v is larger than the distance of u then $v < u$; for example $(1-e, e, I, \ldots) < (P_1, I, I, \ldots)$.

Let M be a matrix satisfying Assumption 3. Fix m, and define the following ordering on quintuples $[u * w(n,m)]_{k,l}$ with $u \in D(M)$, and $k, l < d$ where M is a d by d matrix:

1. If the distance of v is larger than the distance of u then $[v * w(n,m)]_{k,l} < [u * w(n',m)]_{k',l'}$, for every $n, n', k, k', l, l' \leq d$.
2. If the distance of v is the same as the distance of u and $n < n'$ then $[v * w(n,m)]_{k,l} < [u * w(n',m)]_{k',l'}$, for every $k, k', l, l' \leq d$.
3. If the distance of u is the same as the distance of v and v is smaller than u then $[v * w(n,m)]_{k,l} < [u * w(n,m)]_{k',l'}$, for every $k, k', l, l' \leq d$.
4. If $(k,l) < (k',l')$ with respect of lexicographical ordering then $[u*w(n,m)]_{k,l} < [u * w(n,m)]_{k',l'}$, for every u, n.

Fix m. Notice that is an ordering on this set of quintuples $[u * w(n,m)]_{k,l}$ with $u \in D(M)$ and $k, l \leq d$, where M is a d by d matrix.

12. Sets $B_m(M)$ and $Z_m(M)$

We now define sets $B_m(M)$ and $Z_m(M)$.

Definition of set $B_m(M)$

Let M be a matrix satisfying Assumption 3, and m, n be natural numbers.

We will say that quintuple $[u*w(n,m)]_{k,l}$ is in the set $B_m(M)$ if $u*w(n,m)_{k,l}$ is a linear combination over F of elements $v*w(n',m)_{k',l'}$ such that $[v*w(n',m)]_{k',l'} < [u * w(n,m)]_{k,l}$.

Definition of set $Z_m(M)$

Let M be a matrix satisfying Assumption 3, and m, n be natural numbers.

Recall that R is the algebra generated by elements a and b.

We will say that quintuple $[u*w(n,m)]_{k,l}$ is in the set $Z_m(M)$ if there is element $r \in R$ such that:

1. This element r is a linear combination of elements $v * w(n', m)_{k',l'}$ such that $[v * w(n', m)]_{k',l'} \leq [u * w(n, m)]_{k,l}$.
2. r is not a linear combination of elements $v*w(n', m)_{k',l'}$ with $[v*w(n', m)]_{k',l'} < [u * w(n, m)]_{k,l}$.

Lemma 35. *Fix m. Let M be a matrix satisfying Assumption 3. The sets $Z_m(M)$ and $B_m(M)$ are disjoint, that is $Z_m(M) \cap B_m(M) = 0$.*

Proof. Suppose on the contrary, that there is some element in $[u * w(n, m)]_{k,l} \in Z_m(M) \cap B_m(M)$, then there is $r \in R$ which is a linear combination of some elements $v * w(n', m)_{k',l'}$ such that $[v * w(n', m)]_{k',l'} \leq [u * w(n, m)]_{k,l}$. Because $[u * w(n, m)]_{k,l} \in B_m(M)$ then $u * w(n, m)_{k,l}$ is a linear combination of $v * w(n', m)_{k',l'}$ such that $[v * w(n', m)]_{k',l'} < [u * w(n, m)]_{k,l}$. Therefore r is also a linear combination of $[v*w(n', m)]_{k',l'}$ such that $[v*w(n', m)]_{k',l'} < [u*w(n, m)]_{k,l}$, a contradiction with the definition of set $Z_m(M)$. □

Lemma 36. *Let notation be as above. Let m be a natural number and let $R_m(M)$ be the linear space spanned by all elements from R which are a linear combination of elements $u*w(n, m)_{k,l}$ for some $u \in D(M)$ and some n, k, l with $k, l \leq d$ (where M is a d by d matrix). Then the dimension of the space $R_m(M)$ is the same as the cardinality of set $Z_m(M)$.*

Proof. For every element $[u * w(n, m)]_{k,l} \in Z_m(M)$ we assign an element $r = r(u, n, m, k, l) \in R$ satisfying properties 1 and 2 from the definition of the set $Z_m(M)$. Let $Q_m(M)$ be the linear space spanned by these elements. We will show that $R_m(M) = Q_m(M)$. Observe first that if $s \in R_m(M)$ then

$$s = \sum_{(v,n',m,k',l') \leq (u,n,m,k,l)} \alpha_{(v,n',m,k',l')} \cdot v * w(n', m)_{k',l'},$$

for some $(u, n, m, k, l) = [u * w(n, m)]_{k,l}$ and some $\alpha_{(v,n',m,k',l')} \in F$. If we take a presentation of s with $[u * w(n, m)]_{k,l}$ minimal possible, we in addition get $\alpha_{(u,n,m,k,l)} \neq 0$ and $[u * w(n, m)]_{k,l} \in Z_m(M)$.

We will now show that $s \in Q_m(M)$ by induction with respect to the ordering of the quintuples $(u, n, m, k, l) \in Z_m(M)$. Let $[u_0 * w(i, m)]_{t,t'}$ be the minimal quantuple in $Z_m(M)$. By the definition of $Z_m(M)$ we get that $r(u_0, i, m, t, t') = \alpha[u_0 * w(i, m)]_{t,t'}$ for some $0 \neq \alpha \in F$. If s is a linear combination of elements $v * w(n', m)_{k',l'}$ with $[v*w(n', m)]_{k',l'} \leq [u_0*w(i, m)]_{t,t'}$, then s is a linear combination of element $[u_0 * w(i, m)]_{t,t'}$ and hence $s \in Q_m(M)$.

Suppose now that the result holds for all quintuples from $Z_m(M)$ smaller than some quintuple $(u, n, m, k, l) \in Z_m(M)$. We will show that this holds for this

quintuple. Let s be a linear combination of elements $v * w(n', m)_{k',l'}$ with $[v * w(n', m)]_{k',l'} \leq [u * w(n, m)]_{k,l}$. Recall that $(u, n, m, k, l) \in Z_m(M)$ and so

$$r(u, n, m, k, l) = \sum_{(v,n,m,k,l) \leq (u,n,m,k,l)} \beta_{(v,n',m,k',l')} \cdot v * w(n', m)_{k',l'},$$

for some $\beta_{(v,n',m,k',l')} \in F$, $\beta_{(u,n,m,k,l)} \neq 0$ (by the definition of set $Z_m(M)$). Then for some $\alpha \in F$) we get that $s - \alpha \cdot r(u, n, m, k, l)$ is a linear combination of elements $v * w(n', m)_{k',l'}$ with $[v * w(n', m)]_{k',l'} < [u * w(n, m)]_{k,l}$. By the inductive assumption $s - \alpha \cdot r(u, n, m, k, l) \in Q_m(M)$ and hence $s \in Q_m(M)$. It follows that $R_m(M) \subseteq Q_m(M)$. By the definition of $Q_m(M)$ we get $Q_m(M) \subseteq R_m(M)$, hence $Q_m(M) = R_m(M)$. □

13. Main supporting lemma

Let M be a matrix which satisfies Assumption 3, and let notation be as in the previous section.

Lemma 37. *Let M be a matrix satisfying Assumption 3. Let m be a natural number and let $u = (f_1, f_2, \ldots) \in D(M)$. For every $k \leq m$,*

$$u * w(n, m) = \sum_{j=0}^{n} ((f_1, f_2, \ldots, f_k, I, I, \ldots) * w(j, k)) \cdot ((f_{k+1}, f_{k+2}, \ldots) * w(n-j, m-k)).$$

Proof. This follows from Lemma 28 and from the definition of operation $*$. □

Lemma 38. *Let notation be as in Lemma 37. Suppose that u has distance $k + 1$. Then*

$$u * w(n, m) = \sum_{j} ((f_1, f_2, \ldots, f_k) * w(j, k)) \cdot f_{k+1} \cdot w(n - j, m - k).$$

Moreover,

$$w(n - j, m - k) = w(0, t)w(n - j, m - k - t) + \sum_{i=1}^{n-j} w(i, t)w(n - j - i, m - k - t).]$$

Proof. Since u has distance $k + 1$ then $u = (f_1, f_2, \ldots, f_k, f_{k+1}, I, I, \ldots, I)$, where $f_{k+1} \in \{e, P_1, \ldots, P_{\beta'}\}$ and by assumptions on P_i's we have $f_{k+1}e = f_{k+1}$. The first equation follows from Lemma 37. The second equation follows when we apply Lemma 28 to $w(n - j, m - k)$. □

Let M be a matrix satisfying Assumption 3, and let notation be as in Assumption 3. In particular, H' is an algebra generated by matrices from H and an identity matrix, W is the Wedderburn radical of H and $W^s = 0$.

Lemma 39. *Let notation be as above. Let $f_1, f_2, \ldots, f_k$ be matrices from $W \cup \{I - e\}$. Let $f_{k+1} \in H'$. Let $u = (f_1, f_2, \ldots, f_{k+1}, I, I, \ldots)$, and let $m, n > 0$, $m > k+1$, $m > s$. Then $u * w(n, m)$ is a linear combination of matrices of the form $u' * w(n, m)$ and $u'' * w(n, m)$, where $u' \in D(M)$ and u' has distance $k+1$, and where u'' is of the form $(g_1, \ldots, g_{k+2}, I, I, \ldots)$ where $g_1, g_2, \ldots, g_{k+1} \in E$ and $g_{k+2} \in H'$.*

Proof. Observe that f_{k+1} is a linear combination of elements α_i and β_i, where $\alpha_1 \in E = \{E_1, \ldots, E_\beta, 1 - e\}$ and $\beta_i \in \{e, P_1, \ldots, P_{\beta'}\}$ (where notation is as in the definition of the ordering of $D(M)$). Therefore $u * w(n, m)$ is a linear combination of elements $u(i) * w(n, m)$ and elements $u(i) * w(n, m)$, where $u(i) = (f_1, \ldots, f_k, \alpha_i, I, I, \ldots)$ and elements $u'(i) = (f_1, \ldots, f_k, \beta_i, I, I, \ldots)$, for some i. Observe that $u'(i)$ are in the set $D(M)$ and have distance $k+1$. On the other hand, elements $u(i)$ are of the form $(g_1, \ldots, g_{k+1}, I, I, \ldots)$, where $g_1, g_2, \ldots, g_{k+1} \in W \cup \{1 - e\}$ and $g_{k+2} = I \in H'$, as required. □

14. Introducing sets $U_{k,t}$ and $V_{k,t}$

Let $M = \sum_{i=0}^{\xi} A_i a_i$ be a matrix satisfying Assumption 3. Let notation be as in Assumption 3 and in the previous sections. Let $a_{\xi+1}, a_{\xi+2}, \ldots$ be such that $a_1, a_2, \ldots, a_\xi, a_{\xi+1}, \ldots$ is a basis of $A(\alpha)$ (such elements exist by Zorn's lemma). Recall that $a_1, \ldots, a_\xi \in < x >$; hence we can assume that every a_i is either in $< x >$ or in R.

Denote $Q_t = \sum_{(i_1, i_2, \ldots, i_t) \neq (1,1,\ldots,1)} a_{i_1} a_{i_2} \ldots a_{i_t} A$. Let k, t be natural numbers, and define:

$$V = A(k) a_1^t A, U = A(k) Q_t.$$

Recall that M is a d by d matrix. By $T(U)$ we will denote the set of all d by d matrices with all entries in U, and by $T(V)$ we will denote the set of all d by d matrices whose entries are in V. Recall that s is such that $W^s = 0$.

Let $G : A(m) \cap V \to A(m - t)$ be the linear mapping defined for monomials and then extended by linearity to all elements from $A(m) \cap V$ as follows:

$G(w {a_1}^t w') = ww'$, where w is a monomial from $A(k)$ and w' is a monomial from $A(m - k - t)$. We can then extend mapping G to matrices: if M is a matrix with entries $a_{i,j}$ then $G(M)$ is the matrix with entries $G(a_{i,j})$.

Lemma 40. *Let n, m, k, t be natural numbers with $m > k + t, m > t + s$, $t \geq 1$. Let G be defined as before this theorem. Let $u = (f_1, f_2, \ldots) \in D(M)$ have distance*

$k+1$, so $f_1, \ldots, f_k \in E$ and $f_{k+1} = he$. Then $u * w(n,m) = \bar{v} + \bar{u}$ for some $\bar{v} \in V$, $\bar{u} \in U$. Moreover, $G(\bar{v}) = u * w(n, m-t) + s$, where s is a linear combination of elements of the form $u' * w(n-i, m-t)$ for $i > 0$ and where $u' \in D(M)$ has either the same distance as u or larger distance than u.

Proof. Observe that $w(n,m) = \sum_{i=0}^{n} F_i'$ by Lemma 38, where

$$F_i' = \sum_{j=0}^{n-i} w(j,k)w(i,t)w(n-i-j, m-k-t).$$

By the definition of operation $*$ we get $u * w(n,m) = \sum_{i=0} F_i$ where

$$F_i = \sum_{j=0}^{n-i} [u' * w(j,k)] \cdot f_{k+1} \cdot w(i,t) \cdot w(n-i-j, m-k-t),$$

where $u' = (f_1, f_2, \ldots, f_k, I, I \ldots)$. Recall also that $f_{k+1} = he$ for some $h \notin W$.

Observe that

$$w(j,t) = q_j {a_1}^t + u_j',$$

for some $u_j' \in T(U)$ and some matrix q_j with entries in F. Recall that $w(0,t) = {M_0}^t = M^t = (\sum_{i=1}^{\xi} A_i a_i)^t$ and that $A_1 = e$. Therefore $q_0 = {A_1}^t = e$, so $w(t,0) = e{a_1}^t + u_0'$. It follows that $F_0 = v_0 + u_0$, where

$$v_0 = \sum_{j=0}^{n} [u' * w(j,k)] \cdot f_{k+1} e {a_1}^t \cdot w(n-j, m-k-t) \in T(V).$$

Recall that $f_{k+1}e = f_{k+1}$, by assumption on P_i's. By Lemma 28,

$$G(v_0) = \sum_{j=0}^{n} [u' * w(j,k)] \cdot f_{k+1} e \cdot w(n-j, m-k-t) = u * w(n, m-t).$$

Observe now that $F_i = v_i + u_i$, where

$$v_i = \sum_{j=0}^{n-i} [u' * w(k,j)] \cdot f_{k+1} \cdot q_i {a_1}^t \cdot w(m-k-t, n-i-j) \in T(V).$$

By Lemma 20,

$$G(v_i) = u(i) * w(, n-i, m-t)$$

where $u(i) = (f_1, f_2, \ldots, f_k, f_{k+1}q_i, I, I, \ldots)$. By applying Lemma 39 several times we get that $u(i) * w(n-i, m-t)$ is a linear combination of elements of the form $u' * w(n-i, m-t)$ for $i > 0$ and where u' is in $D(M)$ and has distance at least $k+1$, or $u' = (g_1, \ldots g_l, I, I, \ldots)$ for some $l > s$ and all $g_1, \ldots, g_l \in E$. In the latter case $u' * w(n-i, m-t) = 0$ by Lemma 31, hence the latter case can be omitted. Observe now that $\bar{v} = \sum_{i=0}^{n} v_i$, so $G(\bar{v}) = G(v_0) + \sum_i G(v_i) =$

$u * w(n, m-t) + \sum_{i=1}^{n} u(i) * w(n-i, m-t)$ and the result follows (since u has distance $k+1$). $\square$

Lemma 41. *Let m, k, t, n be natural numbers with $m > k+t$, $m > t+s$ and $t \geq s$, $t \geq 1$ (where $W^s = 0$). Let $u = (f_1, f_2, \ldots) \in D(M)$ have distance larger than $k+1$, so $f_1, \ldots, f_{k+1} \in E$. Then $u * w(n, m) = \bar{v} + \bar{u}$ for some $\bar{v} \in T(V)$, $\bar{u} \in T(U)$.*

Let $G : A(m) \cap V \to A(m-t)$ be asdefined and extended to matrices as in Lemma 40. Then $G(\bar{v})$ is a linear combination of elements of the form $u' * w(n', m-t)$ for $n' \geq 0$ and where $u' \in D(M)$ has distance larger than $k+1$.

Proof. Observe that $w(n, m) = \sum_{i=0}^{n} F'_i$ by Lemma 38, where

$$F'_i = \sum_{j=0}^{n-i} w(j, k) w(i, t) w(n-i-j, m-k-t).$$

By the definition of operation $*$ we get $u * w(n, m) = \sum_{i=0} F_i$ where

$$F_i = \sum_{j=0}^{n-i} [u' * w(j, k)] \cdot u'' * \cdot w(i, t) \cdot w(n-i-j, m-k-t),$$

where $u' = (f_1, f_2, \ldots, f_k, I, I \ldots)$ and $u'' = (f_{k+1}, f_{k+2}, \ldots, f_s, I, I, \ldots)$ (as elements of $D(M)$ have distance at most s).

Observe that

$$u'' * w(j, t) = q_j {a_1}^t + u'_j$$

for some $u'_j \in T(U)$ and some matrix q_j with entries in F. Notice also that $q_j = f_{k+1} h$ for some matrix h since f_{k+1} is the first entry of u''. It follows that $q_j \in W$ since $f_{k+1} \in W \cup \{I - e\}$.

Observe now that $F_i = v_i + u_i$, where

$$v_i = \sum_{j=0}^{n-i} [u' * w(j, k)] \cdot q_i {a_1}^t \cdot w(n-i-j, m-k-t), \in T(V)$$

and $u_i \in T(U)$. By Lemma 28,

$$G(v_i) = u(i) * w(n-i, m-t)$$

where $u(i) = (f_1, f_2, \ldots, f_k, q_i, I, I, \ldots)$. Recall that $q_i \in W$. Because $f_1, \ldots, f_{k+1} \in W$, then by Lemma 39 applied several times we get that $u(i) * w(n-i, m-t)$ is a linear combination of elements of the form $u' * w(n-i, m-t)$ for $i > 0$ and where either u' is in $D(M)$ and has the distance larger than $k+1$, or $u' = (g_1, \ldots g_l, I, I, \ldots)$ for some $l > s$ and all $g_1, \ldots, g_l \in E$. In the latter case $u' * w(m, n) = 0$ by Lemma 31, hence this case can be omitted. Observe now that $\bar{v} = \sum_{i=0}^{n} v_i$, so

$G(\bar{v}) = \sum_{i=0}^{n} G(v_i) = \sum_{i=0}^{n} u(i) * w(n-i, m-t)$; the result follows (since each $u(i)$ has distance larger than $k+1$). $\square$

Recall that s is such that $W^s = 0$.

Theorem 42. *Let M be a matrix satisfying Assumption 3. Let $t > s$, and $k, l > 0, m > k+t, m > t+s, n \geq 0$ be natural numbers. Let $u \in D(M)$. If element $[u * w(n,m)]_{k,l} \in Z_m(D)$ then $[u * w(n, m-t)]_{k,l} \in B_{m-t}(M)$.*

Proof. Since $u \in D(M)$ then u has distance $k+1$ for some $k \geq 0$. By the definition of set $Z_m(M)$ there is $r \in R$ such that

$$r = \sum_{(v,n',m,k',l') \leq (u,n,m,k,l)} \alpha_{(v,n',m,k',l')} v * w(n', m)_{k',l'}$$

where $\alpha_{(v,n',m,k',l')} \in F$ and $\alpha_{(u,n,m,k,l)} \neq 0$. Observe that for every $v \in D(M)$ and every n' we can write

$$v * w(n', m) = q(v, n', m) + z(v, n', m)$$

where $q(v, n', m) \in T(V)$ and $z(v, n', m) \in T(U)$, where $T(U), T(V)$ are as in Lemmas 40 and 41. By $q(v, n', m)_{k,l}$ we will denote the k, l-entry of matrix $q(v, n', m)$ (similarly for $z(v, n', m)$). It follows that

$$r = \sum_{(v,n',m,k',l') \leq (u,n,m,k,l)} \alpha_{(v,n',m,k',l')} (q(v, n', m)_{k',l'} + z(v, n', m)_{k',l'}).$$

On the other hand, recall that elements $a_1, a_2, \ldots$ are either from r or from $< x >$, hence by the definition of set U and since $a_1 \in < x >$ it follows that $r \in U$. Since $U \cap V = 0$ it follows that

$$0 = \sum_{(v,n',m,k',l') \leq (u,n,m,k,l)} \alpha_{(v,n',m,k',l')} q(v, n', m)_{k',l'}.$$

We can apply mapping G to this equation. Let W be the linear space spanned by all elements $v * w(n', m-t)_{k',l'}$ with $[v * w(n', m-t)]_{k',l'} < [u * w(n, m-t)]_{k,l}$. By Lemma 40, $G(q(u, n, m)_{k,l}) - u * w(n, m-t)_{k,l} \in W$. By Lemmas 41 and 40, $G(q(v, n', m)_{k',l'}) \in W$, provided that $[v * w(n', m)]_{k',l'} < [u * w(n, m)]_{k',l'}$ (if v has the same distance as u then we use Lemma 40; if v has distance larger than u we use Lemma 41) . Therefore, $u * w(n, m-t)_{k,l} \in W$. This means that $u * w(n, m-t)_{k,l} \in B_m(M)$. $\square$

15. Main result

Let N be a matrix. Recall that by $S(N)$ we denote the linear space spanned by all entries of matrix N; similarly if $N_1, N_2, \ldots, N_k$ are matrices with entries in A then by $S(N_1, N_2, \ldots, N_k)$ we will denote the linear space spanned by all entries of matrices $N_1, N_2, \ldots, N_k$.

Theorem 43. *Let M be a matrix satisfying Assumption 3. For arbitrary c, there is $m > c$ such that $R \cap S(w(0,m), w(1,m), w(2,m), w(3,m), \ldots)$ is a linear space over F of dimension less than $\sqrt{n}$.*

Proof. Recall that, by Lemma 34, $w(0,m), w(1,m), \ldots \in \sum_{u \in D(M), i=0,1,\ldots} Fu * w(i,m)$ for $i = 1, 2, \ldots k$ and $u \in D(M)$ with distance at most s, where $W^s = 0$. It is sufficient to show that for infinitely many m the dimension of the set $R_m(M) = R \cap \sum_{u \in D_m(M), i=0,1,\ldots} S(u * w(i,m))$ is less than $\sqrt{m}$. By Lemma 36 it is equivalent to show that the cardinality of set $Z_m(M)$ is smaller than $\sqrt{m}$ for infinitely many m.

We will provide a proof by contradiction. Suppose, on the contrary, that there is c such that for every $m > c$, set Z_m has more than $\sqrt{m}$ elements.

By Theorem 42, if $[u' * w(i,m)]_{k,l} \in Z_m(M)$ and $[u * w(i', m')]_{k',l'} \in Z_{m'}(M)$ and $m > m' + s$, $m' > 2s$ then $(u, i, k, l) \neq (u', i', k', l')$. It follows because by Theorem 42 $[u * w(i, m')]_{k,l} \in B_{m'}(M)$ and $B_{m'}(M) \cap Z_{m'}(M) = 0$ by Theorem 35, so $[u' * w(i,m)]_{k,l} \neq [u * w(i', m')]_{k',l'}$.

Let m be a natural number. Recall that M is a d by d matrix with entries in $A(\alpha) \cap < x >$. Recall that $\gamma_y(M) = \sum_{i=0}^{t} M_i y^i$, so $w(i,j) = 0$ if $i > tj$.

Let $C_m(M)$ be the set of all tuplets $[u * w(i,m)]_{k,l}$, where $0 \leq i \leq m(s+2)t$, $k, l \leq d$, $u \in D(M)$. Recall that set $D(M)$ is finite. Therefore there is a constant z such that for every m the cardinality of the set $C_m(M)$ is smaller than zm.

We can now choose $m > z^2$ and $m > c + 3s$.

For $q = 1, 2, \ldots, m$ let F_q be the set of elements $[u * w(i,m)]_{k,l}$ such that $[u * w(i, m + q(s+1))]_{k,l} \in Z_{m+q(s+1)}(M)$ where $u \in D(M)$.

Notice that $i \leq m \cdot t \cdot (s+2)$, as otherwise $w(i, m + q(s+1)) = 0$ (since $w(i,j)$ is zero if $i > t \cdot j$ by the construction of $w(i,j)$). Therefore $F_q \subseteq C_m(M)$, for $t = 1, 2, \ldots, m$.

Notice that the cardinality of F_q is the same as the cardinality of $Z_{m+q(s+1)}(M)$, and hence larger than $\sqrt{m}$. By Theorem 42, $F_i \cap F_j = \emptyset$ for any $1 \leq i, j \leq m$. Therefore the cardinality of $\bigcup_{i=1}^{s} F_i$ is larger than $m\sqrt{m}$.

Recall that $F_i \subseteq C_m(M)$ for $i = 1, 2, \ldots, m$. Therefore, the cardinality of $C_m(M)$ has to be at least $m\sqrt{m}$. This gives us a contradiction, since we showed that the cardinality of $C_m(M)$ is smaller than zm, yet we assumed that $m > z^2$. □

We will now prove that Assumption 1 holds for algebras over a field F, where F is the algebraic closure of a finite field.

Theorem 44. *Let F be an infinite field. Let M be a matrix satisfying Assumption 3. For arbitrary c, there is $m > c$ such that $R \cap S(L(M^m))$ is a linear space over F of dimension less than $\sqrt{n}$.*

Proof. We can now apply Theorem 43 to the matrix M to get that

$$R \cap S(w(0, m), w(1, m), w(2, m), \ldots)$$

is a linear space over F of dimension less than $\sqrt{n}$. By Lemma 30, we have

$$S(L(M^m)) = S(w(0, m), w(1, m), w(2, m), \ldots).$$

□

16. Matrices

The aim of the next two sections is to show that, if N is an arbitrary matrix with entries in $A(j) \cap < x >$ for some j, then some power of N satisfies Assumption 3. Here the notation A and R is not related to the similar notation appearing in previous chapters; instead, R denotes a general ring.

Let F be the algebraic closure of a finite field. Let A be a finitely dimensional algebra over the field F. Let $a_1, a_2, \ldots, a_n$ be generators of A. We assign weight 1 to element a_i for $i = 1, 2, \ldots,$. The product of an element with weight m and an element with weight n has weight $m + n$. A linear combination of elements of weight m has weight m. We will say that an element of R is pseudo-homogeneous if it can be expressed as a linear combination of elements with the same weight (notice that the ring R need not be graded). By R^1, we will denote the algebra which is the usual extension of R by an identity element. By $R(n)$, we will denote the linear space of pseudo-homogeneous elements with weight n in R, provided that $n > 0$, and by $R(0)$ we will denote the space $F \cdot 1$ in R^1.

The following Lemma closely resembles Lemma 1 from [1]. However, our ring need not be graded, so we provide a detailed proof using similar methods as in [1].

Lemma 45. *Let R be a simple F-algebra with an identity element. Let $a_1, a_2, \ldots, a_n$ be generators of A. We assign weight 1 to element a_i for $i = 1, 2, \ldots$. Assume that $1_R = \sum_{i=k}^{k'} b_i$, where b_i is a pseudo-homogeneous element of weight i for each i. Then for every peudo-homogeneous element $h \in R$ there exist pseudo-homogeneous c_i of degree i such that $c_i \in \sum_{j=0}^{i-\deg h} R(j)hR(i-j-\deg h)$, and such that*

$$1_R = \sum_{j=t}^{t'} c_i$$

with $t > k'$ and $t' - t \leq k'$.

Proof. Given an element $p = \sum_{i=l}^{l'} e_i$ with e_i pseudo-homogeneous of degree i, and with $e_i \in \sum_{j=0}^{i} R(j)cR(i-j-\deg c)$ with $e_l \neq 0$, let p^* denote the element $e_l(\sum_{i=k}^{k'} b_i) + \sum_{i=2}^{l'} e_i$. Denote $p^* = \sum_i d_i$ with $d_i \in R(i)$; then $d_i \in \sum_{j=0}^{i} R(j)cR(i-j-\deg c)$. Moreover, if $p = 1_R$ then $p^* = 1_R$. Observe that by starting with polynomial p and repeating the $*$-operation sufficiently many times we get a polynomial with the desired properties. $\square$

The following lemma resembles Proposition 1 from [19]. However, our ring is ungraded, so we need to repeat the argument.

Lemma 46. *Let R be a simple F-algebra with an identity element. Let $a_1, a_2, \ldots, a_n$ be generators of A. We assign weight 1 to element a_i for $i = 1, 2, \ldots$. Let*

$$1_R = \sum_{i=k}^{v} p_i$$

with each p_i pseudo-homogeneous of degree i for some k, v, with $v - k$ the minimal possible. Then all p_i are in the center of R.

Proof. The proof is similar to the proof of Proposition 1 in [1]. We will show first that all p_i belong to the centre of R. Suppose the contrary, and let k' be minimal such that $c' = rp_{k'} - p_{k'}r \neq 0$ for some pseudo-homogeneous r (we can assume that r is pseudo-homogeneous, since every element in R is a linear combination of pseudo-homogeneous elements). Since R is simple, then R equals the ideal generated by c'. Notice that c' is a pseudo-homogeneous element. Let c be a pseudo-homogeneous element which is a product of some generators $a_1, \ldots, a_n$ of R and element c', with element c' appearing at least $2v$ times. Such a non-zero element c exists, since a simple ring is prime.

By the previous lemma,

$$1_R = \sum_{j=t}^{t'} c_i$$

where c_i are pseudo-homogeneous and $c_i \in \sum_{j=0}^{i-\deg c} R(j)cR(i-j-\deg c) \subseteq R_i$. Moreover, $t'-t \leq v$. Recall that $c' = -\sum_{i=k'+1}^{v} d_i$, with $d_i = rp_i - p_i r$. Observe that c' is pseudo-homogeneous of degree $k' + \deg r$ and each d_i is pseudo-homogeneous of degree $i + \deg r$. Recall that each c_i is a linear combination of products of elements a_i and element c'; we can substitute at some place in each of these products $c' = -\sum_{i=k'+1}^{v} d_i$. Therefore,

$$c_t = \sum_{i=t+1}^{v-k'+t} f_j$$

for some pseudo-homogeneous f_j, where each f_j is a linear combination of products of some generators $a_1, \ldots, a_n$ of R and element c', with element c' appearing at least $2v-1$ times and each product is pseudo-homogeneous of degree j (so each f_j is pseudo-nomogeneous of degree j). Now if $v-k'+t \leq t'$ we can substitute in this way for element c_t in $1_R = \sum_{j=t}^{t'} c_i$ and obtain

$$1_R = \sum_{j=t+1}^{t'} c_i'$$

where each c_i' is pseudo-homogeneous of degree i and is a linear combination of products of some generators $a_1, \ldots, a_n$ of R and element c', with element c' appearing at least $2l-1$ times. We can do a substitution as above several times to obtain (because $t'-t \leq l$) that

$$1_R = \sum_{j=t'-(v-k')+1}^{t'} c_i''$$

where c_i'' are pseudo-homogeneous of degree i. Observe that $t'-(t'-(v-k')+1) = v-k'-1 < v-k$. This is a shorter presentation than

$$1_R = \sum_{i=k}^{v} p_i.$$

Hence we obtain a contradiction. $\square$

Lemma 47. *Let R be a finitely dimensional simple F-algebra. Let $a_1, a_2, \ldots, a_n$ be generators of A. We assign weight 1 to element a_i for $i = 1, 2, \ldots,$. Then 1_R is a pseudo-homogeneous element.*

Proof. Since $a_1, \ldots, a_n$ generate R, then there are pseudo-homogeneous elements p_i such that

$$1_R = \sum_{i=k}^{l} p_i.$$

We can assume that $l-k$ is the minimal possible. By the previous lemma, each p_i is central. By the Wedderburn-Artin theorem, R is isomorphic to a matrix ring with coefficients from F. Hence, every central element is of the form $\alpha \cdot I$, where I is the identity matrix and α is from F. Then $p_i = \alpha_i \cdot I$, and since $1_R = \sum_i p_i$, then some $\alpha_i \neq 0$. Then $\frac{1}{\alpha_i} p_i = I$ is pseudo-homogeneous. □

Remark Since F is the algebraic closure of a finite field, then for every matrix m there is a natural number $\gamma(M) > 0$ such that $(m^{\gamma(M)})^2 = m^{\gamma(M)}$ is a diagonalizable matrix. Moreover, if all eigenvalues of M are nonzero, then there is a natural number $\beta(M) > 0$ such that $M^{\gamma(M)} = I$, the identity matrix.

To prove this, we need to restrict ourselves to diagonal matrices, where this result holds, and to the Jordan blocks. Let $\alpha I + N$ be a Jordan block with α on diagonal; then N is a stricly uppertriangular matrix, and hence nilpotent. Let p be a characteristic of the field F. Then $(\alpha I + N)^{p^n} = \alpha^{p^n} I + N^{p^n}$; therefore $(\alpha I + N)^{p^n} = \alpha^n I = I$ for sufficiently large n, as required. For some related results see [1].

If $R_1, R_2, \ldots R_n$ are algebras, then elements of algebra $R = \oplus_{i=1}^n R_i$ will be written as $(r_1, r_2, \ldots, r_n)$, with $r_i \in R_i$.

Theorem 48. *Let F be a field which is the algebraic closure of a finite field. Let $R_1, R_2, \ldots, R_t$ be simple finitely dimensional F-algebras. Let $a_1, a_2, \ldots, a_n$ be generators of $R = R_1 \oplus R_2 \oplus \cdots \oplus R_t$. We assign weight 1 to element a_i for $i = 1, 2, \ldots, t$. Then element 1_R is pseudo-homogeneous.*

Proof. We proceed by induction on t. If $t = 1$ then the result follows from the previous Lemma. Assume that $t > 1$ and that the result holds for numbers $1, 2, \ldots, t-1$. Each element a_i can be written as $a_i = (a_i', e_i)$ with $a_i' \in R'$ where $R' = R_1 \oplus R_2 \oplus \cdots \oplus R_{t-1}$ and $e_i \in R_t$. We can apply the inductive assumption to the algebra $R_1 \oplus R_2 \oplus \cdots \oplus R_{t-1}$ with generators a_i' with weight 1. Then the identity element in this algebra is pseudo-homogeneous. It follows that element $(1_{R'}, e)$ is a pseudo-homogeneous element of R for some $e \in R_t$.

Similarly, by the previous Lemma applied to the ring R_n we obtain that $(f, 1_{R_t})$ is a pseudo-homogeneous element of R, for some $f \in R'$.

Observe that, since a power of a pseudo-homogeneous element is a pseudo-homogeneous element, then $(1_{R'}, e)^\gamma$ and $(f, 1_{R_t})^\beta$ are pseudo-homogeneous elements of the same degree for some $\gamma, \beta > 0$. Let α_1, α_t be eigenvalues of matrix f^β; then for any number c the matrix $cI + f^\beta$ has eigenvalues $c + \alpha_1, \ldots, c + \alpha_t$. The field F is infinite, therefore there are $c, c' \in F'$ such that all the eigenvalues

of matrix $M = c(1_{R'}, e)^{\gamma} + c'(f, 1_{R_t})^{\beta}$ are nonzero. By Remark 16, some power of the matrix M is the identity matrix. As M is a pseudo-homogeneous element of R it proves the result. □

We will now prove the following result for rings which are not necessarily semisimple.

Theorem 49. *Let F be a field which is the algebraic closure of a finite field. Let R be a finitely dimensional F-algebra. Let $a_1, a_2, \dots, a_n$ be generators of R. We assign weight 1 to element a_i for $i = 1, 2, \dots$. Then there is a pseudo-homogeneous element $e \in R$ such that $e^2 - e \in W$, and for every $r \in R$ we have $r - er \in W$ and $r - re \in W$, where W is the Wedderburn radical of R (the largest nilpotent ideal in R).*

Proof. Consider the ring $A = R/W$. Then $a_1 + W, a_2 + W, \dots, a_n + W$ are generators of A. We assign weight 1 to element $a_i + W$ for $i = 1, 2, \dots,$. The algebra R/W is semisimple, so by the previous lemma it has a pseudo-homogeneous identity element $e + W = 1_{R/W}$ for some pseudo-homogeneous $e \in R$. Observe that for every $r \in R$ we have $r + W = 1_{R/W} \cdot (r + W) = (r + W) \cdot 1_{R/W}$, hence $r - er \in W$ and $r - re \in W$ for every $r \in R$. In particular $e^2 - e \in W$. □

Theorem 50. *Let F be the algebraic closure of a finite field. Let R be a finitely dimensional F-algebra which is not nilpotent. Let $a_1, a_2, \dots, a_n$ be generators of R. We assign weight 1 to element a_i for $i = 1, 2, \dots,$. Then there is $f \in R$ such that $f^2 = f$ and for every $r \in R$ we have $r - fr \in W$ and $r - rf \in W$, where W is the Wedderburn radical of R.*

Notice then that for every $r \in R$ we have $f(fr) = fr$, because $f^2 = f$.

Proof. Let e be as in the previous theorem. Then $e^2 - e \in W$, and by the remark before Lemma 48 there is $m > 0$ such that $f = e^m$ satisfies $f^2 = f$. Notice also that for every $r \in R$ we have $r - r \cdot e^m \in W$. Indeed, the latter follows because $r - e^m r = (r - er) + (er - e^2 r) + \dots (e^{n-1} r - e^m r) \in W$, since $r' - er' \in W$ where $r' = e^i r$, by assumption. Similarly $r - e^m \cdot r \in W$. □

17. Matrices and noncommutative algebras

Lemma 51. *Let $N = \sum_{i=1}^{\xi'} A'_i a'_i$ where for each i, A'_i is a matrix with coefficients in F and $a'_1, a'_2, \dots a'_{\xi'}$ are elements from A which are linearly independent over F and have the same degree. Let H' be the F-algebra generated by matrices A'_i. Assign weight 1 to every matrix A'_i. Let β be a natural number. Then, for some*

ξ, $N^{\beta} = \sum_{i=1}^{\xi} A_i a_i$ where each a_i is a product of exactly β elements from the set $\{a_0', a_1', \ldots, a_{\xi}'\}$. Moreover, $a_0, a_1, \ldots, a_{\xi}$ are linearly independent over F and $A_1, A_2, \ldots, A_{\xi}$ span the set of matrices that have weight β in H'.

Proof. Observe that distinct products $a_{i_1}' a_{i_2}' \cdots a_{i_\beta}'$ are linearly independent over F, because each of them has the same degree and each of them is a product of elements starting with a or b—the generators of R. Therefore their products are linearly independent over F. Alternatively, it can be proved by induction on β. $\square$

Lemma 52. *Let notation be as in Lemma 51, and let H be an algebra generated by matrices $A_1, A_2, \ldots, A_{\xi}$. Let W be the Wedderburn radical of H, and W' a Wedderburn radical of H'. If $r \in H \cap W'$ then $r \in W$.*

Proof. Observe first that the ideal generated by r in H' is nilpotent, as $r \in W'$. Therefore the ideal generated by r in H is nilpotent, and since W is the sum of all nilpotent ideals in H it follows that $r \in W$. $\square$

Theorem 53. *Suppose that F is a field which is the algebraic closure of a finite field. Let notation be as in Lemmas 51 and 52. Then for infinitely many β the matrix $M = N^{\beta}$ satisfies the following: either $M = 0$ or $M = \sum_{i=1}^{\xi} A_i a_i$, where for each i, A_i is a matrix with entries in F and $a_1, a_2, \ldots, a_{\xi}$ are elements from $A\cap < x >$ which are linearly independent over F and have the same degree. Moreover, there is an element $e \in H$ which is a linear combination of matrices $A_1, \ldots, A_{\xi}$ and such that $e^2 = e$ and $r - er \in W$ and $r - re \in W$, where W is the Wedderburn radical of H.*

Proof. Recall that $N = \sum_{i=1}^{\xi'} A_i' a_i'$. If the algebra H' generated by $A_1', A_2', \ldots, A_{\xi'}'$ is nilpotent then $N^{\beta} = 0$ for some β. Let W' be the Wedderburn radical of H'. If H' is not nilpotent then by Artin-Wedderburn theorem H'/W' is a direct sum of matrix algebras over F (since F is algebraically closed). We can apply Theorem 50 to algebra H'/W to get that there is $e = f$ such that $e^2 = e$ and $r - er \in W'$ and $r - re \in W'$, where W' is the Wedderburn radical of H' and e is a pseudo-homogeneous element of degree β, for an appropriate β. By Lemma 51, matrices $A_1, \ldots, A_{\xi}$ span the linear space of pseudo-homogeneous elements of degree β, hence e is a linear combination of matrices $A_1, \ldots, A_{\xi}$. By Lemma 52, $W \cap R \subseteq W'$, therefore $r - re \in W$ and $r - er \in W$.

We can now take $M = N^{\beta}$. By Lemma 51, $M = \sum_{i=1}^{\xi} A_i a_i$ where for each i, A_i is a matrix with coefficients in F and $a_1, a_2, \ldots, a_i$ are elements from A which are linearly independent over F and have the same degree; moreover, $A_1, \ldots, A_{\xi}$ span

the set of elements that have weight β in H'. By Lemma 51, we can assume that a linear combination of elements $A_1, \dots, A_\xi$ gives element $e = f$ like in Theorem 50.

To show that there are infinitely many elements β with this property, observe that for any natural number $k > 0$ we can apply the same reasoning to $\beta_k = k \cdot \beta$ and $e_k = e^k$ instead of β and e. In this way we will obtain infinitely many $\beta \in \{\beta_1, \beta_2, \beta_3, \dots\}$ satisfying the thesis. $\square$

Corollary 54. *Let notation be as in Theorem 53. Then we can assume that $A_1 = e$ (by using linear combinations of elements a_i instead of elements a_i).*

Corollary 55. *Let A, A', R be as in Theorem 5, and $< x >$ denote the ideal generated by x in A'. Let N be a matrix such that either entries of N are in R or for almost all i entries of N^i are in $A(j) \cap < x >$ for some j; then for some n either $M = 0$ or matrix $M = N^n$ satisfies Assumption 3.*

Proof. It follows from Theorem 53 and from Corollary 54. $\square$

Theorem 56. *Let F be a field which is the algebraic closure of a finite field. Then Assumption 1 holds for F-algebra A.*

Proof. Let N be a matrix with entries in $A^*(j)$ for some j, and such that for almost all α matrix N^α has all entries in $< x >$. Let q be such that N^q has entries in $< x >$, and denote $N' = N^q$. By Theorem 44 applied to $M = N'$, there are infinitely many n such that the dimension of the space $R \cap S(L(N'^n))$ doesn't exceed $\sqrt{n}$. Observe that $S(L(N'^n)) = S(L(N^{q \cdot n}))$ (because operations S and L depend only upon the matrix, not on the way it is presented). Therefore $S(L(N^{q \cdot n}))$ has dimension $\leq \sqrt{n} \leq \sqrt{qn}$. This holds for infinitely many n. $\square$

Proof of Theorem 1. Observe that the algebraic closure of any finite field is countable and infinite. By Theorem 56, Assumption 1 holds for F-algebra A. Then, by Theorem 27, there is an F-algebra Z and a derivation D on Z such that the differential polynomial ring $Z[y; D]$ is Jacobson radical but Z is not nil.

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